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Using children's writing as a window into their mathematical thinking

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USING CHILDREN'S WRITING AS A WINDOW INTO THEIR MATHEMATICAL THINKING

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ABSTRACT

USING CHILDREN'S WRITING AS A WINDOW
INTO THEIR MATHEMATICAL THINKING

by Jill D. Troy

This thesis examines the researcher's use of her fifth grade students' mathematical pen pal correspondence with a group of pre-service teachers enrolled in an elementary school mathematics methodology course as a means of examining elementary school students' mathematical thinking. The research was conducted as an action research project intended to help the researcher examine her teaching and better tailor instruction to the needs of her students. A qualitative perspective was employed in the presentation and analysis of the data.

The researcher found that dialogue sometimes developed between pen pals. Factors encouraging and discouraging dialogue were identified. Examination of the mathematical content of the letters revealed a strong reflection of the topics and problem formats used in class instruction. Teaching strategies used by pen pals in the letters were identified. The implications of the correspondence for classroom instruction, including efforts to empower students mathematically, were examined.

ACKNOWLEDGMENTS

To Sidonia Ender Fisher and Louise DuBreuil Troy, my grandmothers.

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CHAPTER ONE

INTRODUCTION

My students have confused and confounded me with their mathematical work. On numerous occasions I have watched students accurately compute multiplication or division problems but compose story problems which revealed serious confusion about the concepts of multiplication and division. One student offered the question, "My sister has five toes on one foot and six toes on the other. How many toes does she have?" as a story problem for multiplication. Student work such as this led me to wonder what leads students to such thinking. Are they simply guessing? Are they misapplying a rule, but doing so consistently? Are they showing sophisticated insight into alternative methods of explaining a concept?

My Personal Mathematical Journey

My experiences with mathematics as a student have led me to believe that mathematics is full of patterns and connections. Mathematics has always fascinated me and held a special place in my life. My mother had a coffee can filled with buttons for sewing projects. I would amuse myself for hours sorting and categorizing the buttons, observing them carefully and comparing their relative proportions in bar graph-like arrangements. My father played mental mathematics games with my siblings and me. Each time we correctly answered one of his questions we could advance a step up a staircase or bunkbed ladder. He adjusted the problems so that as the eldest child I got the hardest questions. He would give us mental math tips, saying things like, “Nine times seven is easy. It's just ten times seven minus one seven.” We loved the game so much we would play it without him and shared it with many neighborhood friends. On my own, I just saw patterns in the numbers around me. At one point I memorized my dog's license tags by making a mathematical formula out of the numbers. Ten years later I easily remember the home telephone numbers of some of my high school friends because I had created formulas for them as well.

In addition to finding mathematics in the world around me, I was a New Math student. No one told me this at the time, but I recognized the change in mathematics from the textbooks and worksheets I had seen from kindergarten through fourth grade to the activities of the Comprehensive School Mathematics Program[1] in fifth grade. I enjoyed the (Papy) mini-computer activities, Venn diagram logic games, and “magic peanut” negative numbers. All I knew about New Math at that point was that the parents on the *Brady Bunch* television show

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[1] The Comprehensive School Mathematics Program (CSMP) was a program of CEMREL, Inc., one of the national educational laboratories located in St. Louis, MO. It was funded by the National Institutes of Education. A major focus of the program was to develop curriculum materials for kindergarten through 6th grade students. The curriculum was structured as a spiral to ensure continuing exposure to each content area throughout the program. The content included modern mathematics concepts such as the language of arrows, the study of relations, the language of strings and classification, the Papy minicomputer. (Schneider, 1982)

(well into reruns by that point) complained that they could not help their children with their "New Math" homework.

In middle school and high school I participated in the Mathematics Education for Gifted Secondary School Students (MEGSSS)[2] program. The textbook series used, the Elements of Mathematics, was produced by CEMREL and addressed many modern mathematics concepts not covered in traditional secondary school mathematics curriculum. Book titles included *Set Theory* and *Logic*. My real world and school experiences with mathematics led me to believe that mathematics should be challenging, creative, and exciting.

Project Overview

As a teacher I bring this enthusiasm for the patterns, connections, and challenges of mathematics to my classroom. I want to help students develop mathematical confidence and power. Similarly, I expect that careful searching will help me understand the reasons behind patterns or phenomena or my students' mistakes. For this reason, I attempt to look beyond the errors my students make and hope to find a reason. Writing has proven to be a powerful tool for me in the effort to understand my students' understandings and misunderstandings in mathematics. Success in this search for reasons allows

[2] Mathematics Education for Gifted Secondary School Students (MEGSSS) was a program of CEMREL, Inc. as well. The purpose was to prove an instructional program for gifted student utilizing the Elements of Mathematics Program, a curriculum specifically designed for student with superior reasoning ability in mathematics and reading skills (Bodine 1982).

me to help students either extend or remediate their (mis)understandings. It has also helped me feel successful as a mathematics teacher.

This research project details my use of one particular source of my students' writing in mathematics to help me understanding their thinking in mathematics. My fifth grade students wrote mathematical pen pal letters to pre-service teachers enrolled in a university mathematics education methodology course during the fall semester of the 1996-97 school year. The project also examined how I used my students' writing to help me tailor my instruction to their needs. Two questions guided this research:

- What prior knowledge, beliefs, or understandings about mathematics are unveiled in student and pre-service teacher writing?
- What insights into my own teaching are facilitated by the use of students' writings about mathematics?

Teaching Mathematics Today

> The social injustices of past schooling practices can no longer be tolerated. Current statistics indicate that those who study advanced mathematics are most often white males. Women and most minorities study less mathematics and are seriously underrepresented in careers using science and technology. Creating a just society in which women and various ethnic groups enjoy equal opportunities and equitable treatment is no longer an issue. Mathematics has become a critical filter for employment and full participation in our society. We cannot afford to have the majority of our population mathematically illiterate: Equity has become an economic necessity (National Council of Teachers of Mathematics 1989, 4).

Mathematics plays a gatekeeping role in our society. People without clear, complete understanding of the concepts of mathematics and their applications are disempowered in our society. Mathematical understanding is increasingly essential for success in many jobs within the fields of business, finance, health, and defense. Clearly, we need to change the fashion in which we teach mathematics so that the majority of our population is mathematically literate.

The use of manipulatives, exploratory and problem-solving investigations, and concept-based activities advocated by the National Council of Teachers of Mathematics' (NCTM) *Curriculum and Evaluation Standards for School Mathematics* (1989) is meant to provide students with concrete experiences to help them build their conceptual understanding. This approach to mathematics instruction, however, is structurally and philosophically disparate from the traditional show-tell-practice approach experienced by most of today's teachers in their own childhoods.

Teachers who are accustomed to following a mathematics textbook directly, showing students how to compute, explaining a concept or algorithm, having the students practice it, and then testing on this skill, are recreating the patterns of instruction they experienced as students, patterns which have failed to provide access to higher mathematics for women and students of color. This approach often ignores the discussion or reflection component of hands-on mathematical activities advocated by the NCTM and other mathematics reform organizations. The discussion and reflection are meant to assist students in making connections between the activities and the abstractions of more formal mathematics. There are not clear right responses to open-ended questions like, "What patterns did you notice in the fractions you generated using the base rate of 2 for 3?" (2/3, 4/6, 6/9, 8/12, 10/15, 12/18, 14/21, etc.). In discussions with three different classes about this pattern, my students have seen all kinds of patterns, but never have the students *immediately* noted that they could generate the same sequence by multiplying the numerator and denominator by the same number. The patterns students have recognized (such as the repeated addition patterns in numerator and denominator) could be used to eventually help them find this pattern of multiplication, but knowing what to ask next or choosing another activity which may help students explore this question further are tasks which most textbooks do not even suggest. Teachers who depend on textbooks may find this new requirement of teacher insight and judgment difficult. To eliminate this "troublesome" aspect of the new curriculum would eviscerate such a curriculum and make success for new mathematics programs unlikely.

Even pre-service teachers, without ingrained teaching habits, approach teaching mathematics with certain expectations. Carolyn Schiebelhut (1994) found that although the pre-service teachers in her methodology course used manipulative materials in their clinical classroom lessons, very few of them incorporated the use of writing as a tool for teaching mathematics. She states that using children's writing can help teachers recognize when the intended mathematical concepts are not getting through to students. This reflective component of teaching mathematics is often missing from the personal experience and preparation of teachers. Such reflection could help teachers and students to better understand how experiences with manipulative materials affect students' understandings of mathematical concepts.

In an effort to make mathematics a challenging, active, interesting subject, I wanted my students to make connections, notice patterns, talk, manipulate materials, write, question, and generally take control of their own learning while I provided direction and support. I wanted students to see themselves as capable mathematicians and problem-solvers with skills and understanding to tackle any problem and the ability to clearly explain and defend their thinking mathematically. In doing this, I was trying to teach differently from the manner in which I was taught.

Teaching Differently and the Further Changes Ahead

The mathematics textbook series adopted by my school district and provided for my use, Silver Burdett Ginn's *Mathematics*, was overwhelmingly computational. A typical chapter consists of a series of two-page spreads illustrated by one example of the type of problem to be solved on those pages. The sample problem is followed by a page and a half of computation practice problems with an occasional word problem at the bottom of the second page. The district curriculum guidelines similarly list thirty-nine discrete computational skills which fifth grade students are expected to master.

Starting from this base, I sought to make mathematics a challenging, active, interesting subject with activities which would stimulate my students to make connections, notice patterns, talk, manipulate materials, write, and question. I used this textbook as an outline for topics and sought to take advantage of its strength: computation. As a result, I used the text as a source for computational homework problems, classroom warm-up problems, and tests which could be used (in addition to other more thoughtful assessments) as gross measures of student understanding of topics at the end of each unit. The practice problems also worked well for directed skill lessons. This text, however, provided a narrow view of mathematics as arithmetic, a body of algorithms to be memorized and applied.

I needed to create a mathematics curriculum which involved the students actively in the topics. The large number unit, for example, which the class studied during the course of the pen pal letter exchange, began with a discussion of what the students already knew about large numbers, when they used them in their lives, and why we would study them. Using the Know-Want to Know-Learned (K-W-L) format, students identified what they already knew and wanted to know about large numbers. Ira Shor explains this approach of first eliciting student opinions, ideas, and questions about a topic before sharing expertise. "In short, their words and their ideas are the points from which the class begins its critical journey forward" (1992, 29). Beginning with student understandings is also consistent with Paulo Freire's assertion that reading is,

> . . . preceded by and intertwined with knowledge of the world. . . . The understanding attained by critical reading of a text implies perceiving the relationship between text and context (Freire and Macedo 1987, 29).

Mathematics, like reading, is meant to represent reality. Large numbers have been developed to help us express understandings we already have about items in our world. Mathematics instruction, likewise, should help students connect the language and skills of mathematics to the concepts and applications they have already experienced (and continue to experience) in their lives. In class I referred to the list of the life contexts the students had identified in which large numbers were important, connecting them to activities such as using scales and measuring cups to determine whether a bag of birdseed contained a million seeds, calculating whether they might eat a million meals in their lives, and constructing models of 10,000, 100,000, and 1,000,000 blocks to extend the familiar base ten blocks. At the end of the unit we returned to the original question of what the children knew about large numbers and how they related to their lives. We updated the K-W-L chart to include what the students had learned about large numbers. This structure of building on what students already know and connecting new concepts to their lives incorporated a critical dimension into the curriculum (Shor 1992, 41).

In making these changes to the curriculum, I sought to broaden the mathematics curriculum from a focus on merely the skills of arithmetic to include the concepts behind them. Envisioning the traditional mathematics curriculum as a single thread of computational skills, I wanted to add a second thread, conceptual understanding, and twist the two together, creating a stronger rope.

I found strong support for this change in constructivist philosophy and research. In the course of writing about this project I have raised my expectations for elementary school mathematics curriculum even further. I hope to add a third thread, social applications, to this rope. I found support for this critical extension in the work of social constructivists concerned, not only with how people learn, but also with the over-arching purposes and social implications of learning. While I have not yet reached this level of integration in my own classroom, I do have a clearer picture of where I am going.

Using Reflection to Assess Conceptual Understanding in Students

Reflection is essential for helping students at all levels to make connections between activities and the abstractions of mathematics. Putting their thinking into written language not only helps students to clarify their own thinking, but also makes their thoughts more visible to the teacher. Reflection may take many forms including discussion, written responses, and whole class or small group work. Donald Chambers (1993) links reflection to on-going assessment, stating,

> The teacher's priority should be to attempt to understand how the students are thinking rather than to get the students to understand how the teacher is thinking. By constantly thinking about assessment during instruction, teachers can gain knowledge about the thinking of their students and create opportunities to extend each student's understanding. By thinking of instruction and assessment as simultaneous acts, teachers optimize both the quantity and the quality of their assessment and their instruction and thereby optimize the learning of the students (25).

Whatever form the reflection takes, teachers need to better understand what students are thinking about mathematical concepts in order to assess whether the students' thoughts are effectively aligned with those of formal mathematics, pursuing idiosyncratic but correct alternate formulations, or are leading toward a structure which will not demonstrate complete conceptual understanding of a given concept.

Using Writing In Mathematics to Bridge Changing Mathematics Instruction

While many teachers, reform-oriented organizations, and researchers suggest using writing as a tool for teaching mathematics (California State Department of Education 1989; Cook and Craig 1991; Countryman 1992; Ehrich 1994; Gay and Thomas 1993; Kamii 1994; Kamii and DeClark 1989; Kennedy 1985; Miller 1992b; Miller and England 1989; Miller and Hunt 1994; Mumme and Shepherd 1993; National Council of Teachers of Mathematics 1989, 1991, 1995; Sipka 1990; Smith, Kuhs and Ryan 1993; Standera 1994; Szetela 1993; Talman 1990), little research has focused on the benefits of using writing in the mathematics class.

L. Diane Miller (1992a), however, conducted research with secondary mathematics teachers using impromptu writing prompts, brief questions about content which are answered in class, and examined what teachers could learn about their students' understanding of mathematics through writing and how instructional practices are influenced as a result of reading the students' responses. She claims,

> Most of the research on writing in content areas has focused on student benefits. . . . In general, however, there is a dearth of literature addressing the effect which the use of writing in mathematics has on teachers (3).

Teachers in Miller's study found that students' understandings were not always as well developed as teachers assumed them to be. In fact, they found students' understanding and ability to use mathematical vocabulary quite limited.

Pilot Project

A pen pal project was initiated during the 1994-95 school year, pairing my fourth grade class with a class of pre-service elementary education students enrolled in a mathematics methodology course at San Jose State University taught by Dr. Carolyn Nelson. Initially Dr. Nelson and I anticipated that the pre-service teachers would benefit primarily from learning what fourth grade students understood about mathematics and their skill levels as writers.

We paired fourth grade students with pre-service teachers who corresponded over the course of the San Jose State University Fall 1994 semester. This arrangement was repeated during the Spring 1995 semester. Correspondence each semester focused on a specific fourth grade mathematics content area, division during the fall and fractions during the spring. At the end of the semester Dr. Nelson and the pre-service teachers came to my classroom as a group to meet with their student pen pals.

During this visit pen pal groups learned to play division or fraction games. The small groups of fourth graders had opportunities to receive personal attention from their pen pals, while pre-service teachers had opportunities to interact with elementary school students as well as a chance to learn more about the students' understanding. This was also an opportunity for pre-service teachers to recognize the valuable assessment information which could be gleaned through such interactions with students engaged in mathematical activities.

Benefits of the Pen Pal Experience

During the school visit by the second semester group of pre-service teachers, I asked all the participants what they believed they had learned from the experience. I called on volunteers and recorded responses. The fourth graders' most common response was that the project was fun. Writing and solving problems posed in the letters, as well as meeting their pen pals and playing (mathematical) games during the visit, were described as making the experience fun. "Older people can be cool," noted one student. "They can talk about different things with you." Another student noted that the visit was fun, "because in person you could show and do things which take more energy to write about."

The pre-service teachers noted a range of benefits as well. "Fourth grade mathematics content can be fun for all ages" suggested one. Another pointed out that "multiplication and division can be more than memorization;

they can actually be fun." One pre-service teacher noted that it was "important for teachers to be nice because then kids will try harder." Another noted that he had learned how he could incorporate student interests into the questions he formulated.

Some of the pre-service teachers remarked on how the letter-writing exchange had opened their eyes to the level of communication skills of many of their younger pen pals. "Fourth graders can be very articulate in their writing and communication," noted one pre-service teacher. Fourth graders knew more and higher level mathematics than one pre-service teacher had thought. Another echoed the comment that the students were better mathematical communicators than expected. Clearly, pre-service teachers through such reflection gained insights into teaching and student learning as a result of the pen pal experience.

Research Project

The pen pal letter exchange for this research was conducted during the fall 1996 semester between my class of fifth grade students and pre-service teachers who volunteered to participate from multiple sections of the elementary school mathematics methodology course taught by another faculty member. In the past, larger elementary school class size had led to a team letter approach. To facilitate collection of more individualized information about the students, the fifth graders and their pen pals were matched on a one-to-one basis as much as possible. The correspondence was initiated by the elementary school students.

A letter structure (Appendix One) was used to help both groups of correspondents understand the mathematical component of these mathematical pen pal letters. In the pilot work some elementary school students had initially written friendly pen pal letters and shared personal details without including any mathematical content. The structure ensured that both parties understood the role of posing and responding to mathematical problems through the letters. The project is further detailed in Chapter Three.

CHAPTER TWO

REVIEW OF LITERATURE

This chapter outlines the pedagogical perspective of constructivism, which shapes both this project and its analysis, and the role of social activism to empower students mathematically. The second section of the chapter presents current thought about mathematics instruction in a historical context. The third section presents examples of practice-based observations of researchers examining the application of constructivism and social activism to mathematics in the elementary school classroom. The fourth section outlines the characteristics of dialogue. The fifth section examines the use of writing in mathematics instruction and the benefits of writing in mathematics for both students and teachers. Finally, the focus shifts to question posing.

Constructivism and Social Activism

The influence of both constructivism and social activism reflects the progression through which I have moved in my effort to understand learning. I believe that this approach reflects the different emphases at each level of my education which I have incorporated into my current understanding of how people learn. Using Jürgen Habermas' notion of three forms of knowledge (technical, practical, and emancipatory) as a structure, I explain the growth in my understanding of human learning. Patrick Slattery (1995) describes Habermas' critical theory and the three knowledge constitutive interests, stating,

> . . . technical knowledge, which can be measured and quantified; practical knowledge, which is geared toward helping individuals understand social events that are on-going and situational; and emancipatory knowledge, which attempts to reconcile and transcend the opposition between technical and practical knowledge. Emancipatory knowledge helps us to understand how social relationships are distorted and manipulated by relations of power and privilege (201-2).

Although Habermas does not advance these categories as a progression, I see my understanding of learning progressing through stages similar to his categories.

As an elementary and secondary school student, I was immersed in an educational environment which emphasized discrete knowledge and skills. I was expected to learn lists of vocabulary, master mathematical algorithms, and comprehend the facts of the stories I read. Learning was often competitive and skill-based with reading and math groups in elementary school and several levels of courses in middle school and high school. I accepted the curriculum without reflection. I memorized the seven continents and their correct spellings without wondering why one continuous mass of land contained the continents of Europe and Asia. I memorized the notation and progression of square numbers without ever wondering why they were called *square* numbers. My education at this point emphasized skill and informational knowledge which Habermas identifies as technical knowledge.

As an undergraduate I focused my studies on developmental psychology and began working with children in educational environments. This phase of my education focused on ideas about development and interaction. Through the cognitive development work of Jean Piaget I was introduced to the notion that language and thought were related with thought coming before language and with language a way to represent thought. Particularly significant to me was his theory of a developmental progression with its emphasis on an individual student's need to be ready for certain kinds of learning. At the same time I learned about social learning theory which framed learning as strongly influenced by our social environment. Learning to drive was presented as an example of the role of social interaction. We did not stumble upon the proper way to enter a car and drive based on behaviorist principles of receiving reinforcement for certain behaviors. We learned a great deal about driving by watching others drive and by spending time as a passenger in a car. I graduated from college carrying the understanding that we did not enter school as blank slates. Instead we were actively involved in learning within a social context filled with models. I became aware of the social context in which we learned, the role of language and communication in learning, and the necessity of the participation of the learner in the creation of knowledge. At this point, my education focused on the form of knowledge Habermas characterizes as practical knowledge.

As a graduate student I was introduced to the notion of critical postmodernism. I became aware of the pervasive influence of the modernist factory model on education and the technicist approach to schooling. As a fish, I became aware of the water in which I had been swimming throughout my life. Not only the role of individuals and models, but also the role of our culture and its embedded philosophical foundations and politics, emerged as significant forces in the field of education. This forms the heart of Habermas' third form of knowledge, emancipatory or critical knowledge. Awareness of this broader perspective prompted me to try to understand the impact of culture, language, and politics on my work in the classroom and increasingly consider the role of social activism in my teaching.

Within the context of mathematics technical, practical, and emancipatory knowledge are all valuable forms of knowledge. Technical knowledge of the skills of mathematics is needed for computation and applied problem-solving. Practical knowledge is vital for emphasizing the connections between skills and facts and the environment in which they are used. Emancipatory knowledge is essential for examining the implications of mathematics in people's lives. Each of these knowledge forms is appropriate in certain situations. Problems commonly arise when technical knowledge is emphasized to the exclusion of other forms of knowledge, isolating this skill-based, factual knowledge from its applications and implications.

Instead of abandoning all past theories for the notion that we are shaped by our social environment, I held on the belief that the individual had a strong influence on his or her own learning and incorporated the two ideas. Barbara Rogoff (1990), in seeking to integrate theories of psychology with those of communication, anthropology, and education expresses this duality:

> [C]hildren's cognitive development is inseparable from their social milieu in that what children learn is a cultural curriculum: from their earliest days, they build on the skills and perspectives of their society with the aid of other people (190).

I found that constructivism, not universally accepted as an epistemological position (Noddings 1990), provides an image of the learner as an active participant in learning, a constructor of knowledge, offering clear pedagogical implications.

Social constructivism both reinforces this view of the learner as an active participant in the learning process and more clearly articulates the social aspects of learning. Paulo Freire and Donaldo Macedo (1987), for example, admonish that words "should be laden with the meaning of the people's existential experience" and calls for a dialogical relation between teachers and students (35). Freire and Macedo dismiss the "banking," or technical transmission, model of education as an instrument of oppression, a tool for perpetuating inequities in society and encourages a social constructivist perspective where attention is given to the interaction between language and the social environment.

Learning is clearly influenced by the social context in which it takes place. Learners interact with the people, objects, and social structures in their environments as they learn (Rogoff 1990).

The Social Role of Language

The work of Soviet psychologist Lev Vygotsky (in Rogoff 1990) stresses the interaction of language and thought within a social environment. Challenging Piaget's view that egocentric speech preceded social speech, Vygotsky claimed that these self-directed vocalizations of children are internalizations of social speech. Language is a tool, according to Vygotsky, used to mediate the communication of meaning within a social context. Clearly, language is an important facet of communication and learning.

Expanding the Mathematics Curriculum

As a teacher I struggled to incorporate all three of Habermas' forms of knowledge into the mathematics curriculum. Returning to the image of the three threads of the mathematics curriculum which I outlined in Chapter One, I can now label the threads with Habermas' terms. In my effort to make these changes, I sought to broaden the mathematics curriculum from the focus on the first thread of the skills of arithmetic (technical knowledge) to include the second thread of understanding of concepts and context behind them (practical knowledge). The third thread represents the application and social implications of the use of mathematics in our society (emancipatory knowledge). These three threads of human knowledge interest can be braided together to create a single rope. The braid represents the expansion of the mathematics curriculum to which I aspire. While I have not yet mastered this level of integration in my own classroom, I am gaining a clearer picture of where I want to go.

Mathematics as Power

> Mathematics is the key to opportunity. No longer just the language of science, mathematics now contributes in direct and fundamental ways to business, finance, health, and defense. For students, it opens doors to careers. For citizens, it enables informed decisions. For nations, it provides knowledge to compete in a technological economy. To participate fully in the world of the future, America must tap the power of mathematics (National Research Council 1993, 1).

I wear a button on one of my favorite jackets which boldly proclaims, "Math: Power for All!" I chose to teach elementary school because I wanted to help students gain the basic skills which would enable them to pursue their own interests later in school and in life. I wanted to work closely with a group of students developing basic literacy and mathematical skills. This continues to motivate me in my work in the classroom. I have high expectations for all of my students and want them to leave my classroom confident in their skills as readers, writers, and mathematicians. I am well aware of the underrepresentation of females and people of color in higher level mathematics courses and mathematics professions. While I do not expect all of my students to want to become mathematicians or engineers, I want them all to feel that the option is open to them. For students to become mathematically empowered they need more than my desire; they must have both access to the higher levels of mathematics curriculum and the sense that mathematics is relevant to their lives.

Access

Bell Gale Chevigny (1996) discusses how former Student Nonviolent Coordination Committee organizer Bob Moses created the Algebra Project to bring mathematical power to poor students of color around the country. Moses explained the necessity for young students to develop the skills necessary to succeed in algebra and the mathematical challenges beyond.

> As sharecroppers had to become literate to escape plantation serfdom, Moses argues, so children of "criminalized" inner cities now need math literacy to survive. "Unless this generation masters the quantitative information it needs to compete for work created by the new computer technology, we will grow serfs in our cities" (18).

While many of my students during this project year were children of successful, professional parents, others were not. I wanted every student, whether the child of a professor or a window washer, to gain mathematical power. I wanted them to become empowered to choose the life paths which inspired and excited them, not merely the paths they felt they could make it across.

Relevance

Relevance is a significant component of the effort to empower students mathematically. Students need to believe that mathematics is relevant to their lives. Only when they are convinced of its value and relevance to their lives, will they invest the time and energy necessary to develop the skills and concepts needed to succeed mathematically. This problem is widely recognized:

> The filtering effect of the traditional kindergarten through grade twelve mathematics program has produced an American public believing that only a few are good at mathematics and that it is satisfactory for most people to have negative attitudes toward mathematics (and the people who teach and practice mathematics). The next generations of the public are in our schools now. We must work with them to create a place for mathematics in our culture more like the place for music. That is, although only a few are very talented in music, most are good at it, and all can enjoy and participate in it. The beauty, fascination, and usefulness of mathematics make it accessible and important for every student (California Department of Education 1992, vii-viii).

The mathematical pen pal letter exchange was designed to help me better understand how my students thought about mathematical topics and to involve them in a written mathematical discussion with another person. The letter writing format would force students to write about how they found their answers and challenge them to synthesize their understanding of topics by creating their own problems. They would have the chance, through these discussions, to see the relevance of mathematics in the situations presented by their pre-service teacher pen pals. They would also have the opportunity to connect mathematics to their own lives through the problems they created. Mathematical empowerment is an important component of this project.

Current Thought About Mathematics Education

> In reality, no one can *teach* mathematics. Effective teachers are those who can stimulate students to *learn* mathematics. Educational research offers compelling evidence that students learn mathematics well only when they *construct* their own mathematical understanding (National Research Council 1993, p. 58).

When I first began my research I imagined that I would eventually find a long list of the mathematical concepts which students acquire, carefully ordered to reflect the sequence in which they are learned with notes about the best strategies for presenting them to students. I was interested in how students learned mathematics and, clearly influenced by my understanding of cognitive development, expected to find extensive research documenting this concept development.

Finding no such list I wondered how textbook publishers created scope and sequence plans. Were they based on years of common sense analysis of which concepts are more easily grasped? This seemed a very unmathematical, unscientific, approach to be in use by people who write materials to tell students how to go step-by-step through mathematics for most of their elementary and secondary education.

There did exist, however, extensive research about how students learn in general and specifically how they learn about mathematics. This research is a significant force behind the changes I had already seen in mathematics curriculum and instruction, including documents from the National Council of Teachers of Mathematics. Teacher convenience was no longer the guiding concern. Rather, people were attempting to teach in a manner which complemented the manner in which students learn. In considering current approaches to mathematics instruction, it was important to recognize the changes which led up to this point in history.

Historical Perspective: Modernism and Technical Knowledge

Philosophy has significantly shaped mathematics instruction. Epistemology, the branch of philosophy dedicated to the study of the nature of knowledge, has influenced mathematics instruction by creating definitions of knowledge. Definitions of information and knowledge, with their concomitant implications for learning, teaching, and assessment, vary widely depending upon their epistemological foundations. A philosophical examination of the definition and nature of knowledge is essential because these often unspoken beliefs or assumptions shape one's view of learning, and therefore assessment of learning as well.

One of the hallmarks of the modernist world view has been dedication to science. Fueled by rapid growth in the sciences and humanity's understanding of the physical world emerging with the Enlightenment, philosophers came to value information based on experimentation and observation. Following the lead of Descartes and Newton, it appeared that one could measure, manipulate, and control both the natural and social world. Knowledge from this perspective was considered to exist objectively, separate from the knower. This valuing of observable, empirical data over philosophical, reflective, and other ways of knowing led to the epistemology of empiricism and the dominance of logical positivism (Goldin 1990). Knowledge, in this framework, is considered value-free, objectified, and reflective of reality (Slattery 1995, 202).

Applying Empiricism to the Classroom: Behaviorism

Logical positivism in the twentieth century lent support to behavioral psychology. Building upon the foundations of Edward Thorndike in cognitive sciences, B.F. Skinner's work came to focus upon observable changes in behavior as evidence of learning. He interpreted operant conditioning with careful control of reinforcement as the direct cause of learning. "Skinner boxes" demonstrated such learning with chickens and rats learning to push buttons or exhibit specific behaviors in order to earn food rewards. Skinner's work emphasized the decomposition of learning into small discrete behaviors. Clearly defined tasks could be performed by students to demonstrate their learning. Behaviorist and positivist theory guided most educational research from the 1950s through the 1970s with the guiding "metaphor of the classroom as something like a Skinner box . . ." (Erickson 1986, 126).

Technical Knowledge and Mathematics Education

Through the writing of curricularists like Ralph Tyler and Hilda Taba a technical system of instruction was developed and reinforced. This work led to the widespread use of behavioral objectives in the American educational system where observable measurable learning outcomes needed to be "specified in advance, in order that statements of the objectives of instruction satisfy the verifiability criterion of meaningfulness" (Goldin 1990, 36). This focus led many schools to rewrite the objectives of their mathematics textbooks "in behavioral terms, replacing non-operationally-verifiable words like 'understand' with operationally verifiable words like 'solve correctly'" (Goldin 1990, 36). This approach to writing educational objectives reflected the emphasis on the technical knowledge interest and observable behaviors. The application of this style of objectives is inconsistent with the aims of the practical and emancipatory knowledge interests, however. As a result, these interests were often dropped from the curricular agenda in schools.

Cognitive Development, Constructivism, and Practical Knowledge

Piaget's work included an active role for children as constructors of their own learning. The influences of the environment, he maintained, depended upon a child's current stage of development. As children actively worked to master their environments, Piaget contended, they constructed higher levels of knowledge and thought (Cole and Cole 1989, 15). Piaget also "urged a reciprocal relationship between teachers and students, where respect for the teacher coexisted with cooperative and student-centered pedagogy" (Shor 1992, 12). Vygotsky supported this view of the active learner, but went on to emphasize the social and cultural dimensions of the learner's organization of their experience. He posited that,

> Any function in the child's cultural development appears twice or on two planes. First it appears on the social plan, and then on the psychological plane. First it appears between people as an interpsychological category, and then within the child as an intrapsychological category (cited in Cobb, Wood, and Yackel 1990, 126).

Furthermore, Vygotsky distinguished between a student's unassisted and assisted performances.

Vygotsky's Zone of Proximal Development (ZPD) defined "what a child can do with help, with the support of the environment, of others, and of the self" (Tharp and Gallimore 1988, 30). First applied in the context of teaching and testing, this theory clearly identified the significance of the social environment in the learning process. Some have articulated the ideal of teaching as "assistance performance in children's ZPDs" (Tharp and Gallimore 1988, 71). While this form of teaching occurs all the time in the home environment, it is rarely found in schools where both class size and common instructional practices work against this approach to teaching. According to Roland Tharp and Ronald Gallimore (1988), teachers were not taught in this fashion at school, nor were they taught how to teach in this manner.

Constructivism today, according to Ernst von Glasersfeld (1990a), emphasizes an interactive view of the influence of environment on people, where experiences are subjective and personally understood based on our unique life circumstances, experiences, and world view. *Assimilation* (incorporation of new information into existing schema) and *accommodation* (adjusting schema to account for new information), terms used extensively by Piaget, are used to describe an individual's interactions with the environment.

Piaget and Vygotsky's theories reflect the social dimensions of Habermas' form of practical knowledge, emphasizing understanding, the role of people as active creators of knowledge, and the relationship between language and meaning. Nicholas Burbules (1993) describes how Hans-Georg Gadamer similarly challenged the positivistic view of understanding as a clear dichotomy between understanding and misunderstanding, stressing that "understandings are constructed, that all attempts to bridge difference are imperfect translations" (115). Such a constructivist view of knowledge is clearly reflected in the *Curriculum and Evaluation Standards for School Mathematics* from the National Council of Teachers of Mathematics and other guiding reform documents.

Constructivism and Empowerment in the Mathematics Classroom

Currently classroom mathematics pedagogy is significantly influenced by constructivism. Von Glasersfeld highlights the pedagogical implications of constructivism for mathematics education, contrasting it with traditional modeling and practice approach of skills-driven lessons, stating,

> Instead, it becomes a task of first inferring models of the students' conceptual constructs and then generating hypotheses as to how the students could be given the opportunity to modify their structures so that they lead to mathematical actions which might be considered compatible with the instructor's expectations and goals (1990a, 34).

Within a constructivist framework, teachers try to create an environment which will foster or guide student accommodation to alter their models of understanding. From this perspective, teachers are not simply writing information upon a *tabula rasa*, or blank slate, as suggested by English philosopher John Locke in the late seventeenth century. Instead, learners are active participants in their own construction of knowledge.

Theorists debate whether constructivism is an epistemological stance, theory of knowing, methodological perspective, cognitive position, or some combination thereof (Noddings 1990, von Glasersfeld 1990b, Goldin 1990). Constructivists have found widespread agreement, however, that the orientation has implications for learning and teaching in the classroom. Nel Noddings suggests, "constructivism is not a strong epistemological position despite adherents' claims," but underscores "its great strengths as a pedagogical view" (1990, 7).

Believing that learners may construct knowledge based on their experiences has implications for classroom mathematics instruction. Two assumptions which are shared by many advocates of constructivist instruction follow:

- a view of mathematical learning as occurring most effectively through *guided discovery, meaningful application,* and *problem solving*, as opposed to imitation and reliance on the rote use of algorithms for manipulating formal symbols . . .

- approaches to effective teaching through the creation of classroom *learning environments*, encouraging the development of diverse and creative problem-solving processes in students, and reducing the exclusive emphasis on mathematically correct responses (Goldin 1990, 31, emphasis in the original)

This common foundation, however, has not led to a clearly defined model of constructivist teaching in the classroom. Thomas Romberg and Thomas Carpenter (1986) assessed the situation by stating that we currently know more about how children learn mathematics than how to apply this information to mathematical instruction.

While many questions remain about forms of constructivism in the mathematics classroom, there is classroom research supporting the value of a constructivist pedagogy. Both the Missouri Math Project and the Cognitively Guided Instruction project illustrate the benefit of this pedagogy for students. As researchers examine constructivism in the classroom, the concomitant challenges are emerging as well.

Classroom Research Supporting Constructivist Pedagogy

Students are regularly observed choosing to find solutions to mathematical problems using their personal knowledge and strategies despite being taught other methods. "In fact, regardless of mathematics curriculum, community, or country, children (initially) rely on counting strategies to compute sums, differences, and products" (Baroody and Ginsburg 1990, 53). Piaget found that children do not just give up their concepts because adults give them the "correct" method. "Before children will change such beliefs, they must be persuaded that the ideas are no longer effective or that another alternative is preferable" (Confrey 1990, 109). Clearly, it is essential for teachers to understand these guiding concepts which their students use if they want to help change them. Much of the existing classroom research in mathematics has focused on the search for patterns such as these in student thinking about mathematics. The following examples illustrate the findings of researchers examining the impact of applying constructivist pedagogy in classrooms.

The Missouri Math Project: Focus on Development of Concepts

The Missouri Math Project research found that students (in largely traditional mathematics classrooms) performed better in mathematics when their instruction included more time spent on "development," the constructivist element of their mathematics classes. Development was class time spent focused on understanding why algorithms and other mathematical tools work, recognizing how skills are related to concepts, and identifying the mathematical concepts of interest. The middle school component of the research found positive results for student learning when development was emphasized along with a daily ten-minute problem solving component (Romberg and Carpenter 1986). The development and problem-solving emphases of these programs which improved student learning appear similar to the constructivist emphasis on conceptual understanding.

Cognitively Guided Instruction: Focus on Children's Thinking

Recognizing the focus on the individual in Piaget's theory of cognitive development, some social constructivists have looked to Vygotsky's work. Vygotsky shares Piaget's view of learners as active participants in their learning, organizing their experiences, but also incorporates the impact of the "social and cultural dimensions of development" on the learner. Paul Cobb, Terry Wood, and Erna Yackel consider mathematical learning "an interactive as well as constructive activity" (1990, 127). They applied this social constructivist perspective to mathematics instruction through the Cognitively Guided Instruction (CGI) program.

The CGI program was designed to help second grade teachers understand how children think and to facilitate their use of this knowledge in the classroom (Chambers and Hankes 1994). Problem solving was the major goal of instruction and principal instructional strategy. Teachers were encouraged to create problems set in familiar contexts to help use children's prior understandings to forge connections between mathematics and the world outside of school. Teachers were encouraged to use talking in the class to help students articulate mathematical ideas and expose classmates to other solution strategies. Assessment was to be integrated with instruction and used to direct subsequent instruction (Chambers and Hankes 1994). Analysis of the success of the second grade teachers trained in this program found that the emphasis on concepts and problem-solving did not result in lower performance by their students on tests of computational skill (Fennema et al. 1996). Furthermore, they found "strong evidence that knowledge of children's thinking is a powerful tool that enables teachers to transform this knowledge and use it to change instruction" (Fennema et al. 1996, 432).

Challenges of Applying Constructivist Pedagogy

As constructivism has become a significant force in mathematics education in the United States, the challenges of applying this pedagogy in the classroom have emerged. Noddings framed the challenge, claiming,

> Given our premises, we need to get thinking out into the open, to encourage students to conceive their own mathematical purposes and execute their own plans, and to provide situations and objects that may trigger conflict (disequilibrium) and reflective abstraction. How can we do all this with a classroom full of students, and what pitfalls lie in our way (1990, 16)?

Many teachers and researchers are responding to this challenge.

Deborah Loewenberg Ball (1990) identified three simultaneous concerns faced by teachers as they attempted to implement constructivist theories in their teaching. Teachers faced the need to consider the mathematical content of their instruction while also respecting, validating, and building upon student understanding and establishing a classroom community in which students could discuss ideas and come to understandings in groups and individually (Ball 1990). Her second point emphasizes the responsive nature of instruction. All along the way, teachers needed to decide when to directly correct or gently steer students, when to take the class or an individual on a detour, and how to verify student ideas which may not be traditionally of interest in mathematics.

Martin Simon (1995a) advanced a model for constructivist teaching based on his instruction of pre-service teachers. He proposed the Mathematics Teaching Cycle, "a schematic model of the cyclical interrelationship of aspects of teacher knowledge, thinking, decision making, and activity" based on his experience (Simon 1995a, 135). Simon also suggests that teachers applying constructivist tenets predict the path along which their students' learning might proceed. Calling this a hypothetical learning trajectory, Simon emphasized the flexibility of this prediction by the teacher. A hypothetical learning trajectory may be altered by changes in the teacher's knowledge brought about by assessment of student thinking. Within Simon's model such assessment is on-going (1995a).

Leslie Steffe and Beatriz D'Ambrosio responded to Simon's model, agreeing that constructivist beliefs do not "stipulate a particular model of teaching mathematics" (Steffe and D'Ambrosio 1995, 146). They also presented their model of teaching shaped by constructivist beliefs. Simon's response emphasized similarities he saw emerging as theorists pondered models of constructivist instruction. These similarities include: 1) teacher construction of models of students' understanding of mathematics, 2) continuous construction and adjustment of the teacher's knowledge as she interacts with students and they construct knowledge, and 3) emphasis on the importance of a teacher having anticipated or recognized what direction and form the students' potential learning may take (Simon 1995b, 162). These traits appeared in the Cognitively Guided Instruction program (detailed above in the "Cognitively Guided Instruction: Focus on Children's Thinking" section), lending support to the notion that they were characteristics of constructivist instruction in mathematics.

Examples of Emancipatory Interest in Mathematics

The shape of mathematical empowerment, the inclusion of the emancipatory knowledge interest, in the elementary school classroom is starting to emerge. Marilyn Frankenstein has worked to incorporate Habermas' emancipatory interest into mathematics instruction at the college level. Challenging her students to engage in "*critical* discussion about things [they] already know and talk about *uncritically* every day" (Shor 1992, 58, emphasis in original), her students have examined, for example, how the federal government "hides the true size of the military budget by including military costs in non-defense categories" (Shor 1992, 148). In addition to posing problems to raise her students' political awareness, her instruction also incorporates many strategies meant to mathematically and politically empower her students in the classroom. Frankenstein encourages students to codevelop the curriculum by inventing their own math problems (including those used on quizzes). Explicitly teaching her students to teach one another and to work out problems in groups, she helps her students counter the competition and sense of isolation often found in mathematics classes. She also encourages her students to study errors to look for the logic behind them, emphasizing that her students are "thinking beings even when they don't get the answer right" (Shor 1992, 150). Additionally, Frankenstein and others "emphasize student writing about their use of math in their everyday lives" (Shor 1992, 76).

In middle school and elementary classrooms democratic student involvement can also be facilitated through mathematics. Nancy Schniedewind and Ellen Davidson (in Shor 1992) raise student awareness of discrimination by having students collect demographic data about the people they encounter and see portrayed in the media. The students are involved in codeveloping the curriculum by writing materials for class. Ira Shor (1992) claims that this strategy "democratizes authority . . . Coming up with math exercises helps desocialize them from being passive learners isolated from each other by competition" (152).

Across the curriculum, elementary school teachers can incorporate the emancipatory interest into their classrooms by promoting dialogue between students and structuring small-group activities and peer teaching. Noting that there is a strong political component in the selection of the curriculum, Shor (1992) offers a list of questions about the classroom structure to help assess the political nature of the environment. His questions challenge teachers to consider the level of open discussion (versus one-way "teacher-talk"), the amount of student talk (both to the teacher and to peers), the level of student-student cooperation, and the role of students in codeveloping the curriculum in their classrooms. Higher levels of these qualities indicate a more democratic classroom environment. While very few of the examples cited in these discussion are specific to mathematics, these strategies clearly have application in mathematics instruction as methods for creating more democratic classrooms which incorporate the emancipatory interest.

Dialogue in the Classroom

> We engage in dialogical approaches not because they are methods guaranteed to succeed, but fundamentally because we are drawn to the spirit of equality, mutuality, and cooperation that animates them (Burbules 1993, 143).

Dialogue is one way to incorporate the emancipatory interest into the classroom. This approach reflects a critical postmodernist emphasis on the inclusion of "context, historicity, variety. and the constructed nature of all systems of belief and value" (Burbules 1993, 4). Burbules (1993) goes on to claim that dialogue,

> . . . tends toward a decentered and nonauthoritarian view of learning . . . can be supported empirically by reference to contemporary cognitive psychology. Many current studies of learning emphasize that knowledge is structured in memory by "schema," cognitive structures that comprise complex relations of words or concepts (Anderson, 1977; Bransford & McCarrel, 1974; Spiro, 1977). In this view, understanding, or "comprehension," involves incorporating new information into existing schemata an/or altering schemata in light of new information. An important corollary of this model, from the standpoint of learning, is that merely presenting students with new information without adequate attention to their current structures of understanding virtually guarantees that the new material will be forgotten (because a student has no clear associations for it) or misunderstood (altered to fit existing preconceptions) (9).

Dialogue may support the learning process within a social context. Clearly, the role and importance of dialogue is consistent with the constructivist view of knowledge creation as well as the broader social values emphasized by social activism.

Characteristics of Dialogue and Rules Governing It

Burbules (1993) outlined the characteristics of dialogue, noting that not all traits would be present in any given example of dialogue. Dialogues involve two or more people; occur in a climate of open participation; are constituted of alternating statements; are continuous, developmental, exploratory, and interrogative; require the commitment of all parties to sustain it; and reflect reciprocity among the participants in the form of mutual interest, respect, and concern for one another.

Dialogue is similarly a rule-governed exchange. Habermas, in developing his comprehensive theory of communication, outlined standards or truth claims required for communication oriented toward understanding: comprehensibility, trust, sincerity, and rightness (Burbules 1993). Burbules identified a similar set of rules for dialogue: participation must be voluntary and open to active involvement, there must be a mutual commitment to "allow the flow of conversation to be persistent and extensive across a range of shared concerns, even difficult or divisive ones," and reciprocity must exist, allowing the relation to be "reversible and reflexive" (1993, 82). These rules, however, are usually unspoken and recognized only when problems arise in the dialogue. Burbules purposely frames the rules loosely to allow for flexible interpretation based on each unique context, noting "it is impossible to analyze the significance of a move in a dialogue apart from how it is situated in a sequence of moves" (1993, 86).

In addition to rules, Burbules identified categories of moves which constitute dialogues: questions, responses, building statements, redirecting statements, and regulatory statements.

Perspectives in Dialogues

Dialogue does not always take one form. Instead, the perspectives of the participants in the dialogue shape the nature of the discourse. Two significant perspectives identified by Burbules are the participants' conception of dialogue and its relation to knowledge, and their attitudes toward their partner(s) in the dialogue. Participants in a dialogue may take a convergent or divergent view of knowledge in the dialogue, believing, respectively, that "the various positions of the interlocutors are, at least in principle, resolvable into a consensus around the correct answer" or that each statement in the dialogue is "irresolvably plural." Similarly, participants in a dialogue may take an inclusive or critical attitude toward their partner(s). The inclusive attitude means "granting at least a provisional plausibility to what one's partner says," while a critical attitude is "more skeptical, questioning; it emphasizes a judgment about the objective accuracy of the partner's position" (1993, 110-1).

Different Forms of Dialogue

Dialogue, according to Burbules' framework, may be convergent or divergent and simultaneously inclusive or critical. The four possible combinations of these characteristics lead to four forms of dialogue: dialogue as conversation, dialogue as inquiry, dialogue as debate, and dialogue as instruction. The inclusive-divergent form of dialogue as conversation stresses understanding between participants. The inclusive-convergent form of dialogue as inquiry "aims toward the answering of a specific question, the resolution of a specific problem, or the reconciliation of a specific dispute" (Burbules 1993, 116). The critical-divergent form of dialogue as debate has a sharply questioning, skeptical spirit but no goal of agreement. The critical-convergent form of dialogue as instruction is the "leading" form of dialogue in which one party leads the other participant(s) through a series of questions with a specific final understanding as the goal. The correspondence between the pen pals most often took the form of dialogue as inquiry or dialogue as instruction. Burbules notes that more than one form of dialogue may appear in a single example and hybrid forms certainly exist.

Writing in Mathematics Instruction

> We might describe this situation by saying that one of the tasks that a teacher faces is to construct in her or his mind a mental representation that matches the student's mental representation (Maher and Davis 1990, 82).

As an adult who understands the mathematical concepts which I want my students to learn, I often find myself stretching to see experiences, events, or data from the perspective of my students. It is difficult to put aside or unlearn concepts or to look beyond a trait which seems so important. My students' mathematical writing has afforded me the possibility to better understand their thoughts about mathematics. For example, I assigned a problem of the week about the patterns which emerge when you multiply by 10 as homework to my fourth grade class. One of my students who regularly had strong insights into the patterns and concepts of mathematics wrote that after a week of pondering the results, organized into a t-chart, he saw no pattern. I found myself relearning that what is so obvious to an adult may not be clear at all to child. I needed to be careful not to assume that my students would see what I see in a situation. As von Glasersfeld (1990a) states,

> In that case it is of little avail to tell students that they are wrong. Instead, it will in most instances be far more productive for the teacher to try to infer the student's current conceptual structures, no matter how illogical they may seem from an adult perspective. It is only when the teacher has some inkling of "where the student is" that ways can be found to guide the student to make an accommodation so that the student's mathematical responses might be considered more compatible with the teacher's expectations and goals (36-7).

If we believe that students construct knowledge (or at least meaningful knowledge), we need to discover where they are and help them recognize their own learning processes. Teachers need to understand what their students know and use the social context if they are to help students build upon that knowledge (assisting their performance in their Zones of Proximal Development) rather than continuing to use a prescribed curriculum under pressure to "cover" all of the designated content before the school year ends. Much formal mathematics instruction does not consider where children are in the learning process and what they understand. Arthur Baroody and Herbert

Ginsburg (1990) state,

> Too often, a direct-instruction approach overlooks individual readiness, moves too quickly, and thus prevents assimilation of new information. . . . The problem is compounded when new topics are introduced before a child has had a chance to assimilate more basic lessons (59).

Further,

> Research on learning shows that most students cannot learn mathematics effectively by only listening and imitating; yet most teachers teach mathematics just this way (National Research Council 1993, 6).

This experience fosters doubt in the students about their own practical knowledge of mathematics and promotes the idea that there is only one right way to solve any given problem (Baroody and Ginsburg 1990). The research supports the claim that ". . . learning occurs through active participation in a classroom in which the focus of the curriculum is learning, not teaching" (Gutierrez and Meyer 1995, 49).

Writing in mathematics can help classroom teachers meet the need for on-going, integrated assessment. It also provides clear benefits for both students and teachers. These benefits are detailed below.

The Need for On-Going Assessment of Student Understanding

Romberg and Carpenter (1986) conclude their review of research on teaching and learning mathematics by advocating the development of new assessment tools, noting, "We must give up the outdated notion that one can assess the learning of mathematics solely in terms of the ability to produce correct answers" (869). Gerald Goldin (1990) stressed that to assess meaningful learning (as opposed to rote learning) educators must assess whether students make "connections between numerical symbols and non-numerical domains, and [if] they make explicit reference to reasoning processes as well as products" (46). On-going assessment plays a critical part in this movement to connect instruction to student understanding.

> Assessment should be an integral part of teaching. It is the mechanism whereby teachers can learn *how* students think about mathematics as well as *what* students are able to accomplish (National Research Council 1993, 69, emphasis in original)

A clear picture of student understanding of a concept makes planning appropriate instruction for the student simpler. Carolyn Maher and Robert Davis (1990) emphasized the importance of teachers understanding the mental representations which students used in mathematics, stating,

> Teachers' knowledge of children's thinking makes it possible for them to challenge and extend students' thinking and appropriately modify or develop activities for students (90).

Researchers involved with the CGI program concluded that a "major way to improve mathematics instruction and learning is to help teachers understand the mathematical thought processes of their students" (Fennema et al. 1996, 432). Francis Fennell (1991) advocates this use of children's thinking to inform instruction. She describes diagnostic teaching as "an attempt to match the needs of students with appropriate instructional activities" (39). In addition to helping teachers meet this goal of meaningfully assessing student understanding, writing in mathematics can also benefit the students.

Benefits For Students Using Writing in Mathematics

> Good mathematics teachers have learned that writing is both a way of thinking and of communicating. By writing freely about problems, students often can discover solutions. By reporting their methods in writing, students can tell others how to solve problems. By reading what students have written, teachers can gain more informed insights into students' knowledge and possible misconceptions, compared with what teachers learn from seeing only students' mathematical computations or answers (California State Department of Education 1989, 2).

Writing in mathematics gives students ". . .a chance to practice inferring, communication, symbolizing, organizing, interpreting, linking, explaining, planning, reflecting, and acting. Writing helps students make sense of mathematics" (Countryman 1992, vii). Writing to learn offers many benefits for students and encourages students to think about a topic and focus on specific concepts by challenging them to analyze, compare facts, and synthesize. It also gives students "the opportunity to formulate, organize, internalize, and evaluate concepts" (Nahrgang and Petersen 1986, 461). This sometimes leads students to make concepts their own or to deepen understanding of a topic (Miller and England 1989). Joan Countryman (1992) cites additional examples of using writing with her students to help them become more aware of what they do and do not know, connect prior knowledge to the current topic, summarize, raise questions about new ideas, and to construct mathematics for themselves. Patricia Ehrich (1994) states, "Writing about mathematics helps students reflect on their own thinking" (32). Writing after an activity can provide the "opportunity to review, reiterate, and thereby deepened that knowledge" (Wilde

1991, 38). L. Diane Miller (1991) states,

> Because writing leads people to think, improved mastery of mathematics concepts and skills is possible if students are asked to write about their understandings (517).

Writing provides a "glimpse into the metacognitive world of the learner" (Fennell 1991, 39) and can be used to shape instruction to meet student needs.

While writing students may become aware of a problem and try to reconcile the problem or clarify their understanding, become more confident about their thinking, notice more connections between their topic and other mathematical topics, explore a problem, or make great insight into a situation (Ehrich 1994). "Writing about mathematics forces construction of understanding, because we cannot write coherently about something we do not understand" (Talman 1990, 107). Writing serves as a learning aid for students, helping "to focus students' thinking on a better understanding of the subject matter" (Miller 1991, 519).

Fostering Connections With Writing

Writing in mathematics can help students make connections between abstract mathematical symbols and the concrete situations they represent. Manipulative mathematics materials are not inherently mathematical (Brown 1996, Noddings 1990). In the classroom the challenge for learners can be identifying the mathematical concepts within the activities. Teachers have mathematical concepts which they intend to communicate through an activity, but "[f]or children, the mathematics is part of what 'they have to do' and the mathematical component is not always easily disentangled from its embedding" (Brown 1996, 58).

Although Cuisenaire rods, for example, may appear to an adult to be a clear representation of the relative values of the numbers one through ten, this representation is not as clear to students without instruction. My seven-year-old sister illustrated this quite well. I dumped a bag of the multi-colored rods on the floor between us. She picked a few up and began examining them. She quickly pulled out three of the longest rods and purposefully arranged them on the cleared space between us. "What number is it?" she asked me. I considered the overlapping, apparently random arrangement of the rods and made a wild guess. I was wrong so she rearranged the rods and asked me again. I surveyed the new design and thought about all of the tricks like this I had fallen for in the past. Perhaps the answer was one more than the last number anyone had said, I thought, so I guessed that number. I was correct. "How did you know?" she asked me. I admitted that I did not yet understand. After she arranged the rods again, I noticed her hands. They were spread out just beneath her knees, with all ten fingers splayed. I guessed ten and was correct. After two more correct guesses she admitted I had cracked her system. My father, incidentally, had abandoned his efforts to understand her code.

I told my sister that the materials had actually been designed for mathematics. "Oh, I know," she said confidently, reaching for rods. I thought that perhaps she had used Cuisenaire rods in first grade as I had. Instead of arranging the rods neatly end to end or beside each other, however, she made seemingly random piles of rods. There were three rods (apparently length was of no interest to her) in one pile and four in another. She fashioned an addition sign out of two rods and an equals sign out of two more. She put them between the piles and to the right to fashion an equation! Finally, she carefully calculated 3 + 4 by counting on from three and placed a pile of seven rods to the right of the equals sign. She had organized her own mathematical system for using Cuisenaire rods. It bore almost no resemblance to its intended mathematical use, however.

Clearly, children need teacher guidance to help them connect experiences with materials of any sort in mathematics to the concepts which they represent. The power of manipulatives as visible, movable materials with which students may experiment is not realized if students never connect the materials to the concepts they are meant to model. "The data show that children become better thinkers when they are encouraged to do their own thinking" (Kamii 1994, 207). Writing can be used as a method of encouraging students to make some of those connections on their own (asking questions such as "What do you notice about these materials?") or to evaluate how well students have made the connection between the materials and the concept (presenting a task such as "Create a model for an addition equation using Cuisenaire rods. Explain how the rods match your equation.").

Slowing Down to Write

The process of writing may also help students better recognize what they do and do not understand about a mathematical concept by forcing them to slow down their thinking as they write. I have experienced this effect in my writing. I have kept a personal journal for years, writing primarily when overwhelmed by doubt, sorrow, or joy. As I take pen to paper I find that I am able to stop ideas from swirling about it my mind as if they are afraid they will be lost. My hand is painfully slow, unable to keep up with my thoughts. My thoughts, then, are forced to slow down as they wait to be recorded. While my thoughts wait, they meander a bit, making connections and noticing patterns which were not apparent before I began to write.

I think writing has much in common with walking. There are many ways to get from point A to point B. One could drive quickly from one to the other, catching glimpses of the sights between and perhaps reading a street sign or billboard. Or one could hop on a bicycle and notice more details about the environs such as the beautiful gardens in front of some houses. Or one could choose to walk, noticing a broader range of details about each sight one passes. A walker might notice the cat and butterfly the bicyclist would not have seen. One can watch or even speak with people on the way to point B. Writing, like walking, takes more time.

Other, more formal observers, have made had similar insights into writing. Sandra Wilde observes, “Gere (1985) described the value of writing as a thinking tool: by forcing a slowdown in thought processes, it frees the brain to play around with ideas and make new discoveries, more fully to integrate new knowledge" (1991, 39). Finally, the NCTM literature states, "Writing about mathematics, such as describing how a problem was solved, also helps children clarify their thinking and develop deeper understanding" (1989, 26). The strength of writing to help learners reframe their knowledge makes writing a valuable tool for reflection after an activity.

Full Student Participation Through the Use of Writing

Writing about mathematics helps to ensure that all students are participating fully, an important component of an empowering pedagogy. Many researchers have noted this benefit of using writing in the mathematics classroom. "The emphasis should be on all students' communication of mathematics, not just on the more vocal students" (NCTM 1991, 96). Miller and England add that, "It is not possible to talk to every student every day. It *is* possible for every student to write every day" (1989, 308, emphasis in original). Countryman posits that,

> Writing in the classroom means that everyone is active. With talk, whether it be lecture or discussion, only one person is speaking at a time. The rest of the people in the room may participate by active listening and notetaking, but it is easy for some students to disengage, or let the few who dominate the class do all the work (1992, 90).

Miller concurs, stating that,

> While teachers can ask open-ended, thought-provoking questions in class or pose them in the context of small-group activities, only a few students become involved in these discussions. Allowing sufficient time

> for students to formulate and respond to open-ended questions orally can create a lull during which other students can become off-task mentally, if not physically, too (1992b, 6-7).

Because all students are participating by formulating their own written responses, teachers have the opportunity to learn about the thinking of every student in the classroom. This enables teachers to make assessments of the progress of the class as a whole as well as individual students. It is possible that the students will benefit the most from the insight teachers gain into their understanding. "If teachers learn how to become better teachers of algebra through the analysis of students' writings in algebra, then the benefits of improved instruction will follow" (Miller and England 1989, 311).

Benefits for Teachers Using Writing in Mathematics

In addition to these benefits for students, teachers using writing in their mathematics instruction may benefit in numerous ways. Student writing in mathematics can expose the logic or patterns within student errors. A focused analysis of students' mathematical writing can highlight the use of specific mathematical vocabulary used by students when solving problems. These benefits are examined in greater detail below.

Analyzing Student Errors

Student writing in mathematics can reveal not only their understanding of a concept, but also their misunderstandings. Many teachers consider mistakes and misunderstandings the most valuable information they gain from student writings. "They *must* display their mistakes, because those mistakes are the high-grade ore I mine; they are what I use to teach" (Talman 1990, 108, emphasis in original). Teachers should reflect on the effectiveness of lessons, noting, for example, whether a task was successful with a class at promoting reasoning or communication and how the task could be improved for future use (NCTM 1991). Student writing can help teachers reflect upon their teaching. "If several students express a similar misunderstanding, you will have a better idea of what they don't understand, why they are confused, and what you can do to clarify their thinking" (Cook and Craig 1991, 12).

Monitoring Use of Mathematical Vocabulary

Teachers using writing as part of their mathematics instruction may be better able to monitor their students' use of mathematical vocabulary. It is difficult to concentrate on all aspects of a classroom as they happen. The use of specific mathematical vocabulary is one such area. Teachers who have used student writing in mathematics have been surprised to learn how rarely their students use mathematical vocabulary (Miller and England 1989; Miller and Hunt 1994). By using writing as one component of discourse students "learn to use, in a meaningful context, the tools of mathematical discourse -- special terms, diagrams, graphs, sketches, analogies, and physical models, as well as symbols" (NCTM 1991, 34). Having identified the use of mathematical vocabulary as a goal for students, teachers can further enhance the power of writing by encouraging or requiring that students use specific terms in their writing. Misuse of terms will help teachers identify areas of confusion about vocabulary (and perhaps concepts) for students. Along with the focus on vocabulary in written responses, teachers should continue to monitor student use of mathematical language to help them develop their overall ability to communicate mathematically (NCTM 1991).

Our slight personal variations in interpretations of the meanings of words are not significant in many situations, but these variations become more apparent when we communicate about abstract topics (von Glasersfeld 1990a). Mathematical concepts are often abstract, making complete understanding of mathematical vocabulary more challenging. Mathematical vocabulary can also be particularly challenging because it "is heavily based on the use of symbols and attaches specific, and sometimes different, meanings to common words" (NCTM 1989, 214). Teachers who examine their students' use of mathematical vocabulary in their writing often notice this specific problem. Algebra teachers examining their students' writings "saw the need to be very explicit and to provide examples when everyday language is used in mathematics contexts" (Miller 1992a, 335).

The NCTM *Standards* make an additional warning for teachers of the middle grades, stating that,

> Because students tend to overgeneralize earlier meanings, special attention should be paid to their understanding of terms and operations. For examples, previous experiences with whole numbers lead students to view the effects of multiplication as an increase in quantity. This conception must now be modified to include the effects of multiplication on fractions and negative numbers (NCTM 1989, 216).

Students in grades five through eight should become more sophisticated in their ability to communicate mathematics. "But this development cannot occur without deliberate and careful acquisition of the language of mathematics" (NCTM 1989, 78). Using writing as part of the mathematics curriculum can facilitate this goal.

Questioning as a Method of Assessment

Questioning is a powerful assessment tool which can be integrated into the use of writing in mathematics instruction. Questioning should be considered within the broader context of assessment. Assessment may include the use of observation, group problem solving, group presentations and projects, group components on tests, and individual notebooks in addition to more traditional individual paper-and-pencil tasks (Lester et al. 1994). Assessment should be aligned with the goals and content of the curriculum. Assessment should be an on-going, integral component of instruction, with assessment data being used to actively influence instructional decisions (Thompson and Briars 1993).

> Problems and group tasks are not the only means of assessing the integration of students' mathematical knowledge. Written tasks can be used effectively. A written task can consist of multiple subtasks, encompassing various aspects of mathematical knowledge and their investigation (NCTM 1989, 207).

Writing and questioning can meld in an assessment as students write their responses to teacher-posed problems or write their own problems.

<u>Posing Good Questions</u>

While questioning is a powerful method of assessment, different forms of questions elicit different kinds of answers. Questions have been broadly sorted into three categories (Barnes 1990 cited in Vacc 1993). Factual questions involve memory processes only. A review of literature on question posing by teachers found that 80 percent of the 50,000 questions the average teacher poses in a year are factual (Watson and Young 1986 cited in Vacc 1993). The majority of questions which teachers pose in the mathematics classroom require only simple responses (Thompson and Briars 1993). Students need non-factual questions to have the opportunity to make comparisons, question previously learned facts, agree or disagree with peers or the teacher, and to reconstruct what they learn (Vacc 1993).

Good questions can provide valuable learning experiences for students and detailed feedback to teachers. Good questions require students to do more than simply remember a strategy to answer them, create situations for student learning in the process of answering them, and have several acceptable answers (Sullivan and Clarke 1991). This might be extended to include the option of several possible solution paths to one correct answer. Such questions are often characterized as open-ended questions. They not only encourage discourse as discussed before, but also help reach students of all ability levels through the existence of multiple solution strategies or solutions. Students working at a concrete level may often directly model a situation or use a counting strategy to find an answer while classmates use more sophisticated strategies.

Nancy Nesbitt Vacc (1993) cites the use of non-factual questions as helpful for both students and teachers. They give students the opportunity to

> talk about what they know and think; we [teachers] listen to what they are telling us; and we plan instruction based on what we are hearing from, and learning from, our students (91).

Alba Thompson and Diane Briars (1993) concur, stating, "To discover the extent to which students are making sense of the material, questions requiring more elaborate, thoughtful responses must be asked" (19).

Question Styles for Assessing Critical Thinking

Open-ended questions do not all need to follow the formula of presenting a problem situation and asking for a solution. Many other question formats are available for assessing students' critical thinking, such as presenting students with a context for a problem, but omitting the question or a fact necessary to solve the problem. Students might generate the question or fact, and then solve their customized problem. Alternately, after students solve a problem provided by the teacher they may then create and solve a similar or related problem. This strategy is used frequently in Japanese mathematics instruction (Nagasaki and Becker 1993). A teacher might present a problem which the students must communicate *about* without actually solving by outlining the strategy which they would use to solve the problem (Szetela 1993). Finally, a teacher could present students with a problem and an incorrect solution, including a conceptual error, procedural error, or misrepresentation of the problem and ask the students to examine the solution and answer questions about it. All of these options provide opportunities for students to communicate mathematical understandings in writing and "allow students to demonstrate the depth of their understanding of a problem, almost an impossibility with multiple-choice items" (California State Department of Education 1989, 1).

Mathematical Questioning in Japan

The success of Japanese students in international comparisons of mathematics achievement has lead to many examinations of mathematics education in Japan and comparisons with mathematics education in the United States. One study compared the questions asked about addition and subtraction in first grade classes in Japan, Taiwan, and the United States. Asian teachers used more familiar contexts and put more emphasis on understanding problems before solving them when questions which required computation in context were presented. Japanese, and to a lesser degree Taiwanese, teachers placed much greater emphasis on students sharing problem-solving strategies. Discussion and questions focusing on conceptual knowledge, such as "How do you know this is a subtraction problem?" was found very rarely in U.S. classrooms but frequently in Asian classrooms (Perry, VanderStoep, and Yu 1993).

Another examination of mathematics instruction in Japan found that most lessons were highly focused with the teacher acting as a guide, drawing out a variety of student thought during discussions (Nagasaki and Becker 1993). In planning lessons, many Japanese teachers also list students' anticipated responses to the problems which will be posed, anticipating how they would respond to students and their potential misunderstandings. This is somewhat akin to the hypothetical learning trajectory Simon proposes as a guidance for teachers applying constructivist principles (1995a). After a lesson, individual student written responses are collected and analyzed by teachers "to evaluate both the lesson and individual students' performance" (Nagasaki and Becker 1993, 44).

The use of discussion as a major approach to integrating student thought and understanding through sharing solution strategies is what distinguishes Japanese mathematics instruction from most instruction in the United States (Harvin 1993). Both discussion and use of student thought, however, are advocated by recent mathematics reform documents (NCTM 1989, 1991, and 1995).

Problem-Posing

> When problem-posing situates itself in the language and perceptions of the students, their diverse cultures are build into the study. When students see their words and experiences in the problems posed, the power relations of study are allied to their interests. It becomes easier for them to understand the meaning and purpose of intellectual work (Shor 1992, 48).

Shor (1992) outlined the challenges of participatory problem-posing for both teachers and students. Teachers present the subject matter "as a problem for students to reflect on in their own language," while the students are "challenged to go beyond themselves, into a new territory not generated from their backgrounds" (77). This approach, according to Shor (1992),

> has its roots in the work of Dewey and Piaget, who urged active, inquiring education, through which students constructed meaning in successive phases and developed scientific habits of mind (31).

Freire maintained that such problem-posing was possible in all disciplines, including mathematics, stating,

> In a table to be learned by heart, 4 x 4 is one thing; 4 x 4 translated into concrete experience is another; e.g., making four bricks four times. Instead of mechanically memorizing 4 x 4, the pupil ought to discover its relation to something in human life (Freire 1973, 124, cited in Shor 1992, 77-8).

This example underscores the importance of connecting mathematics back to the concrete reality it was designed to represent. Emphasizing this connection between the abstract symbolism (4 x 4) and the concrete reality (making four bricks four times) reinforces the message that humans created mathematical symbols to represent real situations they encountered.

Possible Insights into Student Thinking from Assessment

Questions can be used both to guide discussion (and hopefully student understanding) and to illuminate student understanding of concepts. Clearly teachers using a variety of assessment techniques integrated with assessment are more likely to learn about student misunderstandings. Open-ended questions in particular may also help reveal faulty student reasoning which is producing a correct answer in some circumstances. Consider the problem and student solution strategy presented by Thompson and Briars (1993).

> Four of five dentists interviewed recommended Yukkey Gum. What percentage of the dentists interviewed did not recommend it?
>
> A teacher asked one student who correctly answered 20 percent to explain her solution. She responded, "'Of' means multiply, so I multiplied four times five and got 20 percent!" (22)

Interviews may additionally allow teachers to correct misunderstandings or responses which do not make sense to the teacher or to clarify answers which may not have been clear from written responses (Gay and Thomas 1993).

Assessment should be used in mathematics to help monitor students' improvement, making instructional decisions, evaluation students' achievement, and evaluation programs (NCTM 1995). Teachers have the responsibility of "posing follow-up questions to students who are unable to answer [a question or problem] completely" (Sullivan and Clarke 1991, 17). These students need to be assisted to make the connections or develop the concepts so that they ready for concepts which follow. If students do not have opportunities to develop these basic ideas the problem will be

> compounded when new topics are introduced. . . . Because new topics often build upon previous lessons, the child gets caught in a downward spiral of failure (Baroody and Ginsburg 1990, 59).

Again, it is essential to understand what students are thinking when they make errors. As Walter Szetela (1993) states,

> A solution is not truly evaluated unless the teacher understands what the solver has done and knows whether or not the thinking was appropriate (144).

Summary

Faced with changing beliefs about the nature of knowledge and learning, dissatisfaction with the effectiveness of current mathematics instruction in the United States, and the demands for workers to use and apply mathematical learning in what is becoming known as the Information Age, many advocate a complete reform of mathematics education, incorporating changes in philosophical underpinnings, content, and pedagogy. Making change is difficult, however. Many educators and parents, in particular, are concerned about moving away from what has become a clear-cut system of algorithms and rules with textbooks providing a sequence and pace for arithmetic instruction. The alternative to the tell-show-practice format will be a more complicated approach, requiring teachers to assess and incorporate the understanding of a unique group of students into their instruction. Making change is more difficult when the goal is fundamentally different from the previous experience of both the student and the teacher.

Constructivism provides some direction for teachers attempting to make this change. Clearly writing, dialogue, and thoughtful questioning strategies have a place in mathematics instruction. Their use may help integrate mathematics and the social context. Instead of pretending that students learn individually by soaking in teacher lectures about mathematical concepts and algorithms, teachers can involve students in dialogue with each other and the teacher. Both classroom discussion and writing can be used to facilitate this communication about mathematics. Good use of questions can improve the assessment information collected by teachers. Additionally, active participation facilitated by the use of dialogue and writing in mathematics helps meet the empowerment goals of increasing student participation in mathematics classes. The mathematical pen pal project on which this thesis focuses, attempted to integrate the use of writing, dialogue, and question posing into elementary school mathematics instruction. The writing and exchange of letters between my students and the pre-service teachers were one component of a program intended to employ a constructivist pedagogy designed to empower all students mathematically.

CHAPTER THREE

RESEARCH METHODOLOGY

Research Perspective

Guiding reform documents such as the NCTM *Standards* (1989, 1991, 1995) advocate a teaching style or pedagogy for mathematics which emphasizes student activity and conceptual understanding along with computational facility. Teachers in the classroom now and those preparing to enter need to develop professionally to meet these new challenging expectations for their mathematics instruction.

Challenged to move beyond the narrow focus of mathematics instruction which has allowed "it to be mechanized, made repeatable, so that it is more manageable in a school setting," teachers are now expected, and encouraged, to include conceptual understanding, communication, and active involvement in the scope of our instruction (Brown 1996, 61). One method to achieve this goal of professional development is to conduct action research in our classrooms.

As an action research project, this study is specific to the experience of one class and one teacher. It is an examination of how written reflection can be used to enable students to connect their personal understandings of mathematical material, activities, or algorithms to the standard mathematical concept represented by the material, activity, or algorithm.

Success at both improving teacher understanding of student thinking and application of this understanding to help students further develop their understanding is essential if the current moves to reform mathematics instruction are to be successful. Activities without reflection are very likely to lead to disappointment and failure because students will not make the connections necessary for new curriculum and materials to be successful.

Purpose of the Research

As a practicing teacher I felt that I could best help my students and improve myself professionally by focusing my academic research on my own teaching. Seeing an additional advantage to using my research to help pre-service teachers learn from my class and contribute to my research through this involvement, I chose to conduct qualitative action research.

Secondary mathematics teacher Neil Hunt shared his action research experience focusing on the benefits teachers derive from reading their students' writings in mathematics (Miller and Hunt 1994). I found his experience reflected many of the same concerns and strategies I found during my pilot project and work with writing about mathematics with my class. Specifically, Hunt found that writing required students to "demonstrate a different type of understanding" than required by manipulating symbols and following algorithms (300). Hunt concluded ". . . professional development through action research offers rewards that can affect teachers' practice for the remainder of their careers" (302). This reward will spread exponentially with each future class of students.

The mathematical pen pal exchange was also intended to raise the level of student participation in mathematics. Participation was identified by Shor (1992) as a critical element of an empowering pedagogy.

> Participation is the most important place to begin because student involvement is low in traditional classrooms and because action is essential to gain knowledge and develop intelligence (17).

Students wrote mathematical problems, a practice used by Nancy Schniedewind and Ellen Davidson to empower students (Shor 1992, 150). Further, responding to a pen pal's answers and evaluating and commenting on these responses challenges the students to teach someone else, thus encouraging them to look for the logic behind errors and to appreciate different approaches to the same problem. These objectives are again supported by Marilyn Frankenstein's social constructivist mathematics instruction, advancing "democratic approaches [which] integrate the students' context and thought with a critical appreciation of mathematics and society" (Shor 1992, 150).

Not only does such participation foster learning by action and reflection on that experience, it also supports the development of democracy. Shor (1992) states,

> Only by active learning could students develop scientific method and democratic habits rather than becoming passive pupils waiting to be told what things mean and what to do (18).

Students who experience democratic learning environments during their decade or more of compulsory education may develop expectations for greater participation in their work and social environments outside of the classroom.

Through reading and analyzing written student responses to questions, I hoped to uncover student thinking and understanding about mathematical concepts as the students constructed this knowledge based on their existing understanding of mathematics and their participation in mathematical activities in the classroom. I wanted to use the information I find from this analysis to inform my instruction so I might better meet the educational needs of my students.

Action Research

> Formal models are a product of a series of abstractions and formalizations made by researchers who operate in the context of academic reasoning and attempt to satisfy the current standards of their research community. In contrast, teachers operate in the context of pragmatic pedagogical problem solving in which they have to make on the spot decisions as they interact with their students in specific situations (Cobb, Wood, and Yackel 1990, 132)

My classroom is a complex environment. When students are working on mathematics activities, for example, they are basically all doing something a little different at any given moment. They may be solving the same problem but approaching it differently, or solving different problems. We may be working as a class, in small groups, with partners, or individually. Every moment my students are in class I am responsible for providing them with challenging learning activities. There is very rarely time to stop for philosophical wonderings by myself or to ponder possible responses to a question. I need to respond at the moment. The immediacy of the classroom in which I teach all subjects every day limits the kind of research I feel I could undertake while retaining my primary focus on my students. Even interpretive (qualitative) research which seeks to understand an environment more holistically often uses the model of an ethnographic researcher spending long periods of time in the classroom, observing and taking detailed notes. As the teacher I can rarely take that position because I prefer to be an active participant.

Cobb, Wood, and Yackel (1990) conclude that "beliefs and practices are dialectically related," with change in one effecting change in the other (145). These researchers also believe that "classrooms are learning environments for teachers" (139) as well. I wanted to learn from my research experience and I hoped that my work would directly benefit my instruction. Action research has the potential to help teachers learn from their students, informing "not only teachers, but the field of mathematics education itself" (Reeves 1990, 447). These considerations led me to adopt an action research perspective for this thesis project.

Qualitative Research

Data for this qualitative, or interpretive, research was collected in the form of pen pal letters written by both the elementary school students and pre-service teachers, reflective notes recording impressions of the student writing as it was analyzed, and my own field notes about classroom events as students responded to their pen pals or during related mathematics instructional activities. The analysis of collected documents included a phenomenological component as I sought to understand the students' understandings of mathematical concepts and the role of their writing in order to help me assess their learning and adjust my instruction to their needs (van Manen 1990; Seidman 1991). The anecdotal reports of classroom behavior and discussions, as well as the teacher-written notes about the analysis of the student writings and learning, were also examined for insights into the role of writing in their mathematics classroom (Glesne and Peshkin 1992, Hubbard and Power 1993).

Positivist research on teaching assumes that

> history repeats itself; that what can be learned from past events can generalize to future events -- in the same setting and different settings (Erickson 1986, 129).

In contrast, interpretive or qualitative research presumes that microcultures will differ from one classroom to the next, Frederick Erickson (1986) comments,

> no matter what degree of similarity in general demographic features obtains between two rooms, which may be located literally next door or across the hall from one another" (128).

In this interpretivist tradition, I consider my work specific to the unique combination of students in my classroom. It is meant to be a detailed examination of a specific case.

Constructivism also led me to a qualitative perspective. As Norman

Denzin and Yvonna Lincoln (1994) claim,

> The constructivist paradigm assumes a relativist ontology (there are multiple realities) a subjectivist epistemology (knower and subject create understandings), and a naturalistic (in the natural world) set of methodological procedures (13-14).

My instruction and the analysis of the writing and events reflect my belief that understanding is created differently by each individual within a social context. A qualitative approach supports this goal by examining the specific writing of each student within my specific classroom. Finally, qualitative research complements another goal I have for my project: facilitating change (Salz 1992).

Pilot Project

During the pilot project, my fourth grade students at an elementary school[1] in the San Francisco Bay area corresponded with pre-service teachers enrolled in a mathematics methodology class at San Jose State University during the fall and spring semesters of the 1994-95 school year. I initiated this exchange because I wanted to gain insight into the ways my students come to understand and make meaning in mathematics, thus enabling me to improve my mathematics instruction.

The participating elementary school students were enrolled in a public school in a middle-class community in which many parents worked in management positions in the high technology companies of Silicon Valley. Most of the participants lived in two-parent families and a majority were of

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[1] All names and identifying information about the schools and students has been changed to protect their anonymity.

European descent. Only one student did not speak English as a first language, although more students spoke second languages in addition to English at home.

The San Jose State University pre-service teachers were all earning multiple subject credentials or completing course work to convert out-of-state credentials to California credentials. Some had entered the teacher education program directly after graduation from college. Others were returning to school (often part time) to change careers or reenter the work force. Dr. Carolyn Nelson's mathematics methodology class was selected for the pilot project because she was interested in providing such experiences for her students. In addition, I was participating in a teacher discussion group (The Narrative and Change Group) with Dr. Nelson at the time we began the project. The Narrative and Change group met for four years on a monthly basis to discuss issues related to teaching and learning and the role of narrative in reflection about teaching practice. When I began the prcject I had no plans to incorporate this research into graduate study. The study, however, provoked my interest and was expanded in a graduate class in which I was asked to reflect on my interests in teaching.

Pilot Project Correspondence

The pilot study correspondence was initiated when small groups of pre-service teachers wrote to my class. Their original questions focused on the mathematics activities my students did in class and asked primarily affective questions such as which activities they enjoyed the most. A few groups asked about the use of manipulative materials in class. I explained the terminology to the students before passing out the letters to groups of three to four fourth grade students. Each group read and responded to one letter written by the pre-service teachers. Letters were written in class by both the students and the pre-service teachers. The collections of letters were then exchanged.

Students were encouraged to give examples of class activities whenever possible. Once, while waiting for responses to our letters, I asked students to begin their next letters early by writing about an activity they had just completed. For this activity I read the class the picture book *Sea Squares* (Hulme, 1991). *Sea Squares* provides examples of square numbers being generated using sea-related animals and plants. The square number nine, for example, is presented using a rhyming verse about three clown fish with three stripes each, having a total of nine stripes. I paused often during the reading to ask them if they saw any patterns. I encouraged them to explain their patterns and make predictions about the story which would appear on the next pages. By the end of the book students were predicting stories explaining 9 x 9 = 81 and 10 x 10 = 100.

After finishing the book, a list of the square numbers from 1-100 (1, 4, 9, 16, 25, 36, 47, 64, 81, 100) was created on the board, and I told students were told that these were called square numbers. I asked them if they had any ideas about why they were called square numbers. The students shared their ideas.

No one suggested that the numbers could in any way be represented by squares. The class was given time to explore this question in small groups where they could use any of the class materials they thought would help them. They were specifically urged to experiment with the square tiles, counting out a square number of tiles and experimenting with them.

As the students worked on this activity, I moved from group to group, listening to their discussions and asking them to explain to me what they had found. Most groups had counted out the tiles and were arranging them into rectangles (not always squares). When a group had arranged some of the numbers into squares while arranging others into non-square rectangles, I pointed out this observation and wondered aloud if the non-square rectangles could be rearranged into squares. At the end of the work period the class reconvened to share their findings. Students noted that a square number of tiles could be arranged into a square. I asked what multiplication fact the squares represented and added the equations to the list of square numbers on the board: 1 = 1 x 1, 4 = 2 x 2, 9 = 3 x 3, etc. I asked how the students would find more square numbers. (We used this discussion in later weeks to create an illustrated book of square number stories picking up with 11 x 11).

Students were then asked to write about this experience with *Sea Squares* and their understandings of square numbers in a letter to their San Jose State University pen pals. This would force the students to formulate explanations of the patterns they had seen and worked with while also providing a narrative account of the activity, hopefully allowing their pen pals access to their thinking. This specific example could also give the pre-service teachers an idea of the kinds of activities occurring in some elementary school classrooms.

Pilot Project Data Collection

Data collected for the pilot study included primarily documents and observational notes. The pre-service teachers shared some reflective thoughts about the experience in their final written comments about the experience. Copies of all correspondence between the groups of elementary school students and pre-service teachers were collected along with anecdotal notes taken during the visit. Copies of the letters written by the pre-service teachers and reflections written by the fourth graders were also collected.

The pilot project was repeated with my same elementary school class and a different class of pre-service teachers during the spring semester of 1995. A few changes were incorporated into the project design at that time based on the analysis of the first exchange. During the second exchange the fourth graders and pre-service teachers were encouraged to focus on the topic of division. The fourth grade students were limited to pairs as much as possible for the second exchange to ensure more consistent participation by all students in the class. With larger groups, some of the students had been frustrated by the level of participation possible as they tried to compose one letter for three or four students. Other students chose to simply watch, generating complaints from classmates that they were not helping.

Pilot Project Document Analysis

For this pilot work the analysis was limited to examination of the documentary materials and plans for improvements to the project. A list of reflective questions for both groups was also generated as part of the analysis of the first exchange. I thought that such questions could perhaps be helpful in developing a more complete understanding of what the elementary school students and pre-service teachers learned from the exchange. The reflective questions, however, were not incorporated into the second exchange or the final project.

The instructor of the university course and I organized collection of documentary pen pal materials. I spent time after each exchange reviewing pen pal letters, writing notes about impressions of student understanding and the success of specific questions in eliciting student thoughts while also creating a summary of student thoughts and misconceptions. Erickson explained,

> Write-up stimulates recall and enables the researcher to add information to that contained in the unelaborated, raw notes. Write-up also stimulates analytic induction and reflection on relevant theory and bodies of research literature (1986, 144).

I extend Erickson's comments about the value of thoughtful integration of data and researcher insight to my reflective write-ups after each letter-writing exchange based on collections of student writing and other documentary materials. Analysis at this point was unstructured, allowing me to read the letters and observe classroom reactions to learn more about the process.

Pilot Project Visit

The pre-service teachers first visited the class at the school site on November, 1994. The university students sat with the students with whom they had corresponded and played a fraction game. Each player drew two numbers from an envelope and tried to arrange them as a fraction. For example, a person who drew 4 and 8 could make 4/8 or 8/4. The players each drew a picture of the fraction they chose and wrote a statement comparing the two fractions. A team might find, for example, that 4/8 > 1/6. The player who created the larger fraction received a point for that round.

After observing the class and visitors playing this game, I noticed that most of the students were forming fractions with the smaller number in the numerator and the larger number in the denominator. When I asked one student why he had done that, he told me that you "couldn't have" a fraction with the numerator larger than the denominator! This responses did not shock me because we had looked primarily at fractions less than one, and I knew that the textbook series most of the fourth graders had used in mathematics in the earlier grades had addressed only fractions less than one, simply asking the students to label fractions of wholes. They rarely encountered improper fractions, fractions of sets, or mixed numbers.

I decided to stop the class at this point to issue a challenge. I wanted them to take the next two numbers which they drew and make the *two* possible fractions out of them so that they might draw and compare the two. At the end of the visit, I asked the pre-service teachers to write about what they thought they learned from the correspondence and visit. The fourth grade students were asked to write about how the two fractions they could make from 3 and 8, for example, compared. I wanted to now how they knew which fraction would be larger. I hoped that this experience would open the students' understanding of fractions to both their visiting pen pals and me.

Learning From the Pilot Work and a Similar Project

In revising the pen pal project I attempted to consider the activities occurring in both the elementary and university classrooms. Based on the pilot project work, I decided to move away from the group and pair letter approach and assign each student an individual pen pal. I hoped that this change would allow me to monitor each child's writing more closely and ensure full participation by all students, especially students who were weaker in mathematics. I observed that some of these mathematically less confident students would minimize their involvement and participation when composing group letters, letting classmates solve the problems and plan responses for the pen pals. In an effort to ensure that all participants understood the expectation for writing problems and responding to problems, I also formalized a structure for the letters and provided the pre-service teachers with a copy of this suggested outline for each letter (Appendix One). During the final project I reviewed the letter structure with my students each time letters arrived. The suggested letter format included four sections: personal information, answers to problems presented in the pen pal's letter, evaluation of the pen pal's solutions to the problems presented by the writer in the last letter, and presentation of new problems for the pen pal to solve. This format necessitated another shift in the correspondence. In the pilot work I had encouraged the elementary school students to write about mathematics activities from class related to a specific topic and to answer their pen pal's questions. Based on my experiences of: 1) asking my students to write their own mathematical problems in class, 2) hearing students' positive responses to receiving individualized questions in their pen pal letters, and 3) reflecting on the literature on question posing, I chose to focus the mathematical writing of the final project, not on descriptions of classroom activities, but on creation of, solutions to, and evaluation of mathematical problems.

After completing the pilot pen pal letter exchanges, I found an article about a similar pen pal letter exchange between pre-service teachers and fourth and fifth grade students (Fennell 1991). The project was organized by a university instructor who wanted her students to focus on diagnostic teaching, making attempts "to match the needs of students with appropriate instructional activities" (Fennell 1991, 39). The students and pre-service teachers exchanged letters on a weekly basis. The pre-service teachers kept files on their math pals, completing a profile of the student's interests, attitude toward mathematics, math concept and skill development, and problem solving abilities. Francis Fennell suggested a number of items for consideration by those undertaking similar projects. She recommended that researchers require the elementary school students to write out solution plans for the answers they provided to the pre-service teachers' questions, to have pre-service teachers specify the type of problem solving response (e.g., individual or partner work) they were seeking, and to assess the mathematical interests and attitudes of the elementary school students before beginning the letter exchange (Fennell 1991). The first recommendation was implemented during the pilot project. The second recommendation, I believed, would be addressed by giving the pre-service teachers more clear expectations for letters writing. The third suggestion, however, focused on students' attitudes toward mathematics. I decided not to make this an area of emphasis for my research although the topic did arise in some of the correspondence. I chose instead to focus on the students' creation of, solution to, and evaluation of mathematical problems.

Final Project

During the final project, my fifth grade students from an elementary school in the San Francisco Bay area corresponded with pre-service teachers enrolled in a mathematics methodology class at San Jose State University during the fall semester of the 1996-97 school year.

Project Participants

The participating elementary school students were enrolled in a public school in a middle-class community in which many parents worked in management positions in the high technology companies of Silicon Valley, at a local university, or in other professional positions. Most of the participants lived in two-parent families and a majority were of European descent. Over 85% of the students spoke English as their first language. Other students in the class spoke Spanish, Japanese, or French as their first language. During the pen pal project the class was composed of sixteen boys and eight girls.

The San Jose State University pre-service teachers were all earning the multiple subject credential or completing course work to convert out-of-state credentials to California credentials. Some had entered the teacher education program directly out of college, while others were returning to school (often part time) to change careers or reenter the work force. (Throughout this document, I will refer to these participants as both "pre-service teachers" and "university students.") Because Dr. Nelson was not teaching the course during the research period, the final project correspondence involved university students from another instructor's mathematics methodology classes. The participating professor chose not to incorporate the pen pal correspondence into the mathematics methodology curriculum, however. She made voluntary participation in the project one way to fulfill a class requirement of observing elementary school students. This ensured that all of the participating pre-service teachers were interested in the project. This removed the advantages, however, of whole class participation for pre-service teachers. Since the project was not a shared experience for all of the university students in the class, the ability to discuss the experience as a class and the pressure for individuals to write letters outside of class by a given date further limited participation. There were more than enough volunteers, however, to ensure that each of my fifth grade students could write to a single pre-service teacher.

Project Correspondence and Data Collection

The final project correspondence was initiated when my students wrote to the pre-service teachers. The one-to-one pairing of pre-service teachers and elementary school students met my goal for improving the project structure based on the pilot project experience. Initially, there were more pre-service teachers than elementary school students, so some of my students wrote combined letters to two pen pals. When pre-service teacher pen pals did not submit letters, however, I was obliged to reassign pen pals. The letters from the pre-service teachers were collected by the participating professor, and I picked them up when I was on campus for another course. Similarly, I delivered student replies when I came to campus. The mathematics methodology instructor would distribute the letters in class.

The elementary school students wrote their letters in class. The letter-writing project was a required part of the curriculum. Only the work of students whose parents granted permission for their participation in the research, however, were analyzed for this project. Permission was received from twenty of the twenty-four students in the class.

The pre-service teachers wrote letters outside of class. Most of the pen pals wrote to each other four times and received four replies during the course of the project. Some elementary school students, however, received more replies because they had two pen pals. Others received fewer letters because the pre-service teacher pen pal did not submit a reply on time. Copies of all correspondence between the groups of elementary school students and pre-service teachers were collected along with anecdotal notes taken during the visit and while reading through the pen pal letters.

The Project Visit

An attempt to invite all pre-service teachers pen pals to visit the classroom was made by telephone. It proved difficult, however, to arrange a date and time which worked for all of the students. In the pilot project the pen pals had all been enrolled in the same section of the mathematics methodology course which met during the day. This enabled Dr. Nelson to organize one class meeting as a field trip to visit the pen pals. The pre-service teachers participants in the final project, in contrast, were enrolled in two different sections, both meeting after school ended for the elementary school students. Participation in the pen pal project was voluntary so not all of the university students in each section had pen pals. Furthermore, many of the pre-service teachers in the final project worked during the day and were unable to arrange time to visit. The visit was disappointing for both me and the elementary school students who hoped to meet their mathematics pen pals.

Project Document Analysis

I completed preliminary analysis of the correspondence between four of the students and their pen pals as part of my work for a university mathematics course. I searched for patterns or themes in the writing of the elementary school students as I examined these four cases. The complete collection of correspondence was examined in collaboration with a group that included Dr. Victoria Harper and fellow graduate students. We searched for interesting examples, patterns, and themes within the on-going exchange. We made notes individually and shared our observations, separating comments about the elementary school students and pre-service teachers but looking for connections within and between the groups.

Based on this preliminary work, I developed an outline of categories which I expected to support and expand by reading each letter again individually. The categories initially included: Establishing an On-Going Dialogue, Analysis of Questions, Affective Findings, Mathematical Content, and Application of Information. I sorted the correspondence into pen pal groups so that I could read the correspondence between pen pals chronologically. As I read I made note of specific examples which fit into my existing categories, created numerous subcategories, and added one category: Analysis of Teaching Strategies. These findings are detailed in Chapter Four.

Summary

The design of this mathematical pen pal project reflects my commitment to constructivist mathematics instruction, mathematical empowerment for all of my students, and the use of writing in mathematics. Both groups of correspondents were expected to write at least three mathematical problems for one another in each letter. The letters contributed to the creation of a mathematics curriculum guided by constructivist principles. The problems within the letters could ideally provide meaningful problem-solving opportunities for the elementary school students. The interaction between pen pals could provide guidance, especially for the elementary school students, in making sense of the situations in the problems. Composing mathematical problems also challenged the participants to apply their understanding of a subject to a situation. Writing problems also moved students beyond their traditionally passive role of constantly solving problems. This change was intended to raise the students' level of participation by helping to develop the curriculum of the class and to bolster their sense of personal mathematical power. By corresponding with distant pen pals, my students had a new and different audience for sharing thinking about mathematics. They could not rely on personal interaction to fill in any gaps in the communication as they could in class. The written format encouraged them to express their understanding of mathematics as clearly as possible. The written record of their thoughts not only challenged them to learn to articulate their own understandings, but also gave me insight into this understanding. Reading their letters gave me another source of information about their understanding which I could consider along with their performance solving problems. Writing, then, provided benefits for the students who were challenged to master the language needed to articulate their understanding and provided benefits to me in my effort to provide challenges which would advance their understanding (or provide clarification for misunderstandings).

CHAPTER FOUR

DATA PRESENTATION AND ANALYSIS

Wading through the stacks of pen pal letters, themes and patterns did indeed emerge which I have organized into six sections. The first section of this chapter presents the affective dimensions of the pen pal correspondence. The second section examines the establishment of dialogue within the correspondence. The third section details examples of dialogue in the letters. The fourth section examines the mathematical questions and their contents, including computational and word problems, question types, and mathematical topics addressed. The fifth section examines the teaching strategies revealed in the letters. The final section considers the implications of the pen pal correspondence for the classroom. In keeping with the qualitative, classroom-specific nature of this action-research project, the data and analysis are presented in tandem.

Affective Dimensions

I specifically chose not to include the assessment of attitudes towards mathematics as a focus for this pen pal project. Letter-writing, however, encourages personal engagement and many affective findings emerged from the correspondence. Nicholas Burbules (1993) identified this link, explaining, "In fact, it is impossible in the context of the dialogical relation to treat cognitive and affective topics separately" (35). Clearly, the affective dimensions of the correspondence would emerge despite my original intention not to focus on them. Personal engagement emerged in the forms of personal connections between pen pals, encouragement, positive attitudes about writing and positive attitudes towards mathematics.

Personal Connection

Both the elementary school students and their pre-service teacher pen pals openly shared details of their lives, quickly establishing a personal connection between writers. They shared about their families, hobbies, and favorite flavors of ice cream. The suggested letter format encouraged pen pals to share personal information in the first paragraph of the letter. This seemed to help students begin their letters easily. They knew about their own lives even when they were unsure about a mathematical topic or question. The earlier letters had a greater emphasis on personal information. In later letters, pen pals had explained the basics of their lives and wrote less about themselves. Burbules (1993) explains that these simple questions and answers about personal lives, ". . . may help create an interactive pattern of talk and may help to set the tone for a relation of mutual interest and trust" (87).

During the course of the correspondence many of the pen pal pairs exchanged photographs. In the past I received a few copies of each student's school portrait for classroom use and almost every student would send a picture. This year I did not receive any student photographs and fewer students remembered to bring them from home to include with a letter. Some creative pen pal partners adapted to the situation, however, by asking pen pals to draw pictures of themselves, making predictions about what the pen pal looked like, or writing descriptions of their appearances. Some pen pals also sent one another small gifts such as pogs, candy, or paper fortune tellers (an origami-style folded paper toy). Candy was a particularly popular gift sent to the elementary school students in letters exchanged right before and after Halloween. Most of the pen pal partnerships included some example of a positive connection between the partners.

Encouragement

Many of the pen pal letters included examples of one party providing encouragement or positive feedback to the other pen pal, contributing to the positive atmosphere of the correspondence. One elementary school student commented that although her answers seemed correct, he was "no mathematician." His pre-service teacher pen pal responded, "You answered my questions right - so you can call yourself a mathematician!" Another called her elementary school pen pal a "math whiz." Both the elementary school and university students wrote encouraging comments as part of their evaluations of the pen pal's answers to the last set of questions. The younger students often gave their pen pals letter grades (almost all A's and A+'s) and one drew smiling faces beside her problems. The recipient of the smiling faces wrote a note thanking her pal for the encouraging smiles, saying they "gave me confidence and motivation." Both elementary school students and pre-service teachers frequently praised one another, describing their efforts as "good work" or "great work." This positive feedback between pen pals helped establish a positive attitude toward the correspondence and a friendly tone within individual letters.

Positive Attitude Toward the Correspondence

Both groups indicated positive feelings toward the pen pal project. The elementary school students asked in class when they could expect responses. When letters were distributed in class, students were excited and shared details from their letters with one another. After taking some time for this activity, the class would quiet as students began writing their replies. One student wrote to his pre-service teacher pen pal, "I agree a math pen-pal is pretty cool." Students who did not receive a reply were disappointed. One student insisted on writing again to his pen pal although she had not replied, stating,

> Why did you not write back? I waited and never got it. Do you not want to be my Math Pen-Pal? If not I don't know why.

His pen pal apologized for her lapse in her next letter. When the pre-service teachers were contacted individually by telephone to invite them to visit the class, many expressed their appreciation for and enjoyment of the pen pal project.

Positive Attitude Toward Mathematics

Quite a few students mentioned their attitudes toward mathematics during the course of the pen pal correspondence. Almost every reference to liking or disliking mathematics was accompanied by a comment about performance in mathematics. A correlation between the two emerged. Both elementary and pre-service teachers who said that they liked mathematics also said that they believed they were good at it. One elementary school student and one university student made the connection, saying that they both liked mathematics *because* they were good at it.

Similarly, the students who wrote that they did not like mathematics noted that they were not good at it. Two of the pre-service teachers wrote that they had not liked mathematics when they were in elementary school but added that they now found mathematics much more comprehensible and enjoyable. One pre-service teacher wrote, "I think I decided I could never like it [mathematics]. Now I am learning math all over again in a new way and I really enjoy it." Success in mathematics seemed to be related to attitudes towards mathematics for both groups of students.

Establishing an On-Going Dialogue

One theme to emerge from the group examination of the correspondence was the presence of a dialogue in some of the pen pal relationships. Some of the pairs had managed to establish a friendly correspondence in which pen pals responded to, and built on, each other's questions, answers, and ideas.

The interests of a pen pal may have been incorporated into problems, for example. Many of the pre-service teachers, in particular, were able to tailor their writing to the real world of an elementary school student, using a vocabulary and writing style which the younger students understood easily.

Dialogue for This Project

Dialogue is “a conversational interaction directed intentionally toward teaching and learning” (Burbules 1993, x). For this project the conversation was written, not oral, however. Paul Ricoeur (1981) defines text as “any discourse fixed by writing” (145). The texts of the mathematical pen pal letters constitute the discourse examined here. Many of the characteristics outlined by Burbules (reported in Chapter Two) appeared in the written exchanges between the pre-service teachers and elementary school students. These exchanges involved two or more people. Because of the written exchange, this project ensured that the participants made alternating statements. Participation for the pre-service teachers was completely voluntary (with the unfortunate result that some withdrew from the project during the exchanges). The elementary school students, however, were required to participate as part of their regular mathematics participation. Many of the letters evidenced a continuous, interrogative exchange between the participants as well as their commitment to the mathematical conversation. Finally, most of the participants demonstrated reciprocity in the forms of interest, respect, and concern for one another. Ricoeur (1981) concisely identifies dialogue as “an exchange of questions and answers" (146). The pen pal letter exchange clearly involves an exchange of questions and answers. Ricoeur (1981) also identifies a significant characteristic of text: ". . . writing preserves discourse and makes it an archive available for individual and collective memory" (147). This aspect of the letters made possible the careful examination of the discourse.

Broadening Understandings Through Dialogue

> [The commitment to dialogue is] the bond that joins two (or more) persons in the cooperative pursuit of knowledge, agreement, or personal understanding (Burbules 1993, 19).

Participants in a dialogue may arrive at a new, shared understanding. The presence of correct answers in many mathematical situations, however, does not remove the possibility of true dialogue. Dialogue can and should continue as both parties continue to explain their thoughts and perspectives and to question one another's understanding. While one answer may emerge as the correct answer, dialogue may enable participants to understand many different ways to frame and approach a problem. One partner may simply learn from the other as well. Burbules (1993) explains,

> While a broadly egalitarian commitment and mutual respect ought to frame our pedagogical outlook, these should not obscure the ways in which some participants clearly do stand to benefit from an opportunity to learn from (not only *with*) others who know, understand, or can do things that they themselves cannot (22, emphasis in original).

The idea of mutually valid perspectives on the same topic is apparent, too, when considering multiple solution strategies for a single problem and problems with multiple solutions. In the first case, two people may approach the same problem differently and come to the same correct answer. One pre-service teacher gave her pen pal the problem,

> Find the pattern of the following numbers and tell me how you get the answer.
> 1, 3, 7, 15, 31, 63, 127, ___, ___, . . .

Her pen pal replied,

> So what I did is I tried doing this problem different ways and I chose the most reasonable way. The most reasonable answer is for # 1 you add one more number than the number already is and add those numbers together and you get the next number written on your paper. Example: 3, 7   3 + 4 = 7.

Her pre-service teacher pen pal was clearly surprised by this approach.

> Your answer was right. Congratulations! However, I was confused at your explanation the first time I read it. Though your explanation was different from mine, it's reasonable. I guess I want you know how I explained the pattern. For every number, I multiplied by 2 and then added 1 to get the next number. For example, (1 x 2) + 1 = 3; (3 x 2) + 1 = 7, and so on.

This problem solving experience led both participants to learn another way to find the next numbers in the series. In this case, dialogue led both the elementary school student and pre-service teacher to a broader understanding of the series and the mathematics within it. Both had "cracked the code" of the series, but they had done so differently.

In another case, two people solving the same problem came to different, but equally correct, solutions. One elementary school student asked her pen pal if she expected to eat a million meals in her lifetime and explained that she had already solved the problem for homework. Her pen pal provided a detailed explanation of her answer. The elementary school student replied,

> You got the correct answer! There are actually a few different answers depending on how old you put [you expected to live to be]. In mine I had 88 years, but you forgot the leap years!

Her pre-service teacher pen pal replied, "Oops! Stupid me. I did forget the leap years!" In that same letter the pre-service teacher answered a problem about knitting scarves for fifty years and was careful to account for the leap years in her solution. Clearly, problems which have multiple correct solutions open possibilities for productive dialogue. Each party's understanding of a topic may be *broadened* by the dialogue. One university pen pal articulated this position, writing to her pen pal, "[Y]ou and I can learn from each other."

Empowerment Through Dialogue

In the examples cited above, the pen pal correspondence provided the opportunity for both the elementary school student and the pre-service teacher to teach (and learn from) one another, an aspect of a critical mathematics curriculum according to Marilyn Frankenstein (1983). Through the pen pal letters, participants had opportunities to teach, be taught, and to evaluate responses. In her work with college students, Frankenstein sought to create these same opportunities for her students, asking them to teach each other in an effort to counteract the disempowering messages of traditional mathematics teaching methods. Shor (1992) claims that Frankenstein also emphasized evaluation by making

> error itself a theme of study in class, by asking students to figure out the logic of errors they or others make, to see themselves as thinking beings even when they don't get the answer right (150).

The pen pal letter exchange was designed to facilitate these same goals for the participants. By posing problems, answering the problems they received, and commenting back to their pen pals on the correctness of their responses, students had opportunities to teach, learn, and evaluate. "Students can also learn a lot about posing problems by evaluating the clarity, the difficulty, and the interest, of other students' and teachers' problems," Frankenstein explains (1983, 333).

Examples of Dialogue

In many letter exchanges the discussion between the parties illustrated the continuous, developmental, and interrogative aspects of a dialogue. Both parties made the commitment to follow through on problems and showed respect and concern for on another throughout the exchange. This commitment to dialogue is illustrated as two pen pals discussed a problem the elementary school student had posed in his second letter, "If 80 kids wanted too [sic] go around the equator how long would they have to be?" His pen pal solved his problem and then continued,

> We <u>know</u> that no kid could stretch 311 miles, so let's make your question a little more real. Each week we write we can get closer to a solution. I found out how many miles the equator is long (24,902). You find out how many feet are in a mile, then we can get closer to a solution of how many

> kids it would take to stretch the length of the equator. What will be our next step?

The elementary school student replied,

> There are 5,280 feet in a mile. The next step to find the answer is how many kids are in a mile. What's the next step?

His pen pal replied that they now needed to find how many feet each student could stretch. She suggested,

> I think 4 feet is not a big stretch hand to hand. You need to divide 5,280 by 4 and that will us the amount of kids it would take to stretch a mile. Then, you need to tell me what our last step is in our question of kids around the equator.

Her suggestion included a sketch of person with arms outstretched, indicating that the arm span measured four feet. The elementary school student replied by solving 5,280 ÷ 4 and explaining, "1,320 kids fit in a mile. The next step is you have to multiply that by the number of miles that are around the equator." His pen pal enthusiastically answered,

> Yes! We did this problem step by step and came up with an answer! Sorry, this [the final step of multiplying the number of students per mile by the number of miles around the equator] is up to you because I'm at school and don't have the data I need to figure this out.

The dialogue between these two pen pals included this continuing, cooperative effort to solve one problem over many letters. This example reflects many of the factors which encourage the development of dialogue in a pen pal exchange. The problem evolved when the pre-service teacher chose to elaborate upon a problem introduced by the elementary school student, showing her interest in his problem and confidence that they could do more with it mathematically. Throughout their letters both pen pals used language which was easily understood and attempted to make their mathematical problems appropriately challenging for one another. These two participants clearly trusted and respected each other to find the needed information, perform the appropriate calculations, and move the process along to the next step. They both maintained their focus on the topic, continuing to work on the problem over the course of many letters. Both parties were clearly eager to find the truth and respected the other's views. Dialogue, however, could develop in a different situation and have a different focus.

The dialogue between another elementary school student and her pen pal had multiple foci during the course of their correspondence. The first few letters focused on algebra. This topic was initiated by the elementary school student who wrote in her first letter, "Some of my favorite concepts are geometry and algebra (although I don't know much of it!)." Her pen pal replied,

> I am very impressed by your interest in both geometry and algebra! You must be a very smart young lady. When I was your age I really enjoyed geometry and I think I would have enjoyed algebra also if I had been better at it. Maybe you could help me enjoy algebra more by helping me understand it better.

The elementary school student responded with details about what she liked about geometry and asked, "What did you not understand in algebra? (I don't know much algebra myself, but what I do know I like.)"

Her pen pal attempted to explain, writing,

> Algebra, like geometry, has formulas but it was difficult for me to apply the rules and formulas to algebra problems that looked different than the problems I had worked on or memorized. . . . For example, if my homework problem was 24 x n = 96 and my test question was 96 ÷ n = 24 I couldn't do it because, to me, it wasn't the same kind of problem with the same answer of 4. It is really hard to explain. I hope you can kind of understand.

In her next letter the elementary school student pen pal wrote,

> In math we have been learning (you'll love this!) algebra. I think it's very fun. For me it was easy to understand that 44 ÷ n = 4 and 4 x n = 44 meant that n=11.

In the same letter the elementary school student introduced a problem which became the focus of the dialogue.

> You are a sea captain and you want to visit the New World. Your queen gives you $100,000. It costs $10 to hire 1 person. It costs $15 for 100 lbs. of food (1 month for 1 person). You must include yourself. You plan to take one year. How many men can you hire?
>
> P.S. Algebra might help.

The elementary school student drew a smiling face beside the post script comment. Her pre-service teacher pen pal filled a page working on the problem. She finished by writing,

> I have $90,040.00 left over, but that is not enough to hire and feed another person. . . . Can you show me how you would work out this problem. I would like to see your thought process. The problem didn't look right to me because it seems there is too much money left over.

The elementary school student responded to this request for help with a detailed explanation.

> I figured out what was wrong with your answer. Your main problem was the cost of the food issue. A package of 100 lbs. of food is $15. 100 lbs. of food is 1 month for 1 person. 1200 x $15 [an equation the pre-service teacher had used] would mean that each pound is $15. This means your 1 person, 1 month is 100 times too large.
>
> This is how I got my answer:
> 1. $15 x 12 = $180
> 2. $100,000 - $180 = $99,820
> 3. $10 + $180 = $190
> 4. $99,820 ÷ 190 = 525.36842
> 5. 525, with a little to spare
>
> 1. This is how much it costs for 1 person's food.
> 2. Get yourself out of the way.
> 3. The cost of each person [hired].
> 4. See how many men you can afford.
> 5. My answer.

This dialogue clearly followed a different course from the first example. Instead of emphasizing cooperative problem-solving, this dialogue emphasized sharing and teaching. Many of the same factors, however, were involved in creating the dialogue. Again, the pen pals quickly established an atmosphere of respect and trust. The pre-service teacher felt comfortable sharing her misgivings about algebra. The elementary school student respectfully asked about her pen pal's interests in both geometry and algebra and shared her own. When the explorer problem was introduced the pre-service teacher felt comfortable admitting her doubts about her solution and asked her pen pal for assistance. The elementary school student responded supportively, diagnosing her pen pal's error and providing a complete explanation of the solution. Both parties use comprehensible language and when it became clear that a mathematical problem had caused one pen pal to stumble, the other pen pal provided assistance.

Many of the other pen pal pairs provided examples of continuing dialogues in their correspondence as well. Often the topics did not extend through the entire course of the pen pal exchange, however, but ended after a few letters to be replaced with another topic. In contrast, some pen pal pairs had very little connection develop through the letter exchange.

Factors Encouraging Development of Dialogue

Dialogue requires that the participants treat each other respectfully. This respect has been illustrated above as the mathematical pen pals wrote encouraging comments to each other, supporting their efforts in mathematics. The friendly tone of the letters was also fostered by trust between pen pals. Some pen pals felt comfortable enough with each other to admit that they needed help with or clarification of a problem. After filling a page with her attempts to solve a collection of related problems, one elementary school student wrote,

> I'm sorry. I didn't understand the problems. Could you go through the problem in your letter and teach me how I could find the answer?

Her pen pal responded quite positively, clarifying information for the problem, organizing a series of more focused questions to help lead her to the main question, and providing a copy of the questions with answers. She replied,

> Hopefully the way I rewrote the problems will be understandable. There are many steps to the problems, but now they are broken out. I know they are very advanced, but try not to be overwhelmed when you look at them. Just do one step at a time. Don't feel bad if you need to look at the solution.

In the same letter this pre-service teacher admitted she did not understand a problem from her pen pal and asked for hints. Clearly the two trusted each other enough to reveal their need for help and responded encouragingly to such requests.

Another factor encouraging dialogue between pen pals was the incorporation of a partner's interests into a response. One pre-service teacher mentioned that she loved ice cream. In his reply, her pen pal wrote a word problem for her involving ice cream cones. Many pre-service teachers also incorporated a pen pal's interests into the mathematical problems they wrote. One pre-service teacher suggested, "Since we both like soccer let's do math problems on soccer!" Another wrote, "Since you have 104 bottle caps in your collection, let's use that number." Even when problems were not as specifically tailored to student interests, they usually reflected the lives of children. The problems for the elementary school students were contextualized in sports, shopping, and other familiar activities. Inclusion of these details made it immediately clear to each party that the partner was indeed reading letters carefully and responding. This tailoring of the curriculum to student interests by some of the university pen pals indicated their efforts to recognize and build upon the "generative themes" of the younger students' lives, an important step towards establishing a critical mathematics education (Frankenstein 1983).

Many pen pals built upon the ideas suggested by the partner in the last letter, strengthening the connections between the letters. This happened in almost every pen pal pair as personal information was shared as participants responded directly to a partner's request for information or offered parallel information. Less often, pen pals would build upon each other's mathematical problems. One pre-service teacher built a question upon her own answer to her pen pal's question about how many meals she expected to eat in her lifetime. After explaining her solution strategy and answer, she added, "If I miss two or three meals a week, how many meals less would that be?" Other pre-service teachers noted a student's use of a specific topic or question format and responded in kind. One elementary school pen pal explained triangular numbers to her pen pal. Her university pen pal responded with the question, "If you had a triangular number and the base had 20 units, what would that number be?" Again, these connections between letters sent the message that the pen pals were carefully attending to each other's letters and writing personal responses, illustrating the interest and respect between the pen pals. These personalized, building responses underscore the continuous, developmental nature of the exchange between many of the pen pal pairs.

Another factor in successfully establishing dialogue between mathematical pen pals was the use of language and problems appropriate to the elementary school student's understanding. Most of the pre-service teachers had no problem communicating clearly with their pen pals. It did not appear that they needed to make any special effort to restrict or simplify their vocabulary. Matching problems to a student's mathematical skill level, however, was more challenging.

Many pen pals chose to tackle this problem directly and asked their partner for feedback about the difficulty level of the problems. "Do you want harder problems? Or are those OK? Let me know," wrote one pre-service teacher to her pen pal. Another wrote, "In your next letter, please tell me if this kind of problem is too easy for you so I can adjust my math problem to fit your level." While the pre-service teachers seemed to put more thought into making the problems appropriate for their pen pals, some of the elementary school students did consider this. One student wrote to his pen pal, "If any of these [mathematical problems] are too hard you don't have to do it."

Most of the adjustments to match mathematical problems to a pen pal's skill level took place more subtly. A frequently used approach was to take the problems that the elementary school student wrote and create slightly harder similar problems in response. For example, one elementary school student asked what n would be for the equation $350 + 40 = n$. His pen pal responded by asking for the value of n in $n + 123 + 428 = 600$.

Another pre-service teacher appeared to recognize that her pen pal was writing only computational problems. Accordingly, she responded with a couple of computational problems and a single word problem connected to a sport he had mentioned.

Overall, pen pals seemed to be following the advice, "Better to challenge than to bore." The problems were, on occasion, too difficult for an elementary school student to solve alone. Answering the letters in class, however, provided opportunities for assistance from classmates, reference materials, or the teacher.

The creation of dialogue in mathematical pen pal letter exchanges was encouraged by many factors. Establishing an atmosphere of mutual respect and trust between pen pals was an important component. Similarly, incorporating the interests of the partner into problems or building new problems upon those received from a partner helped illustrate this interest and respect while establishing connections between the letters and the writers. Responding to one another's problems also provided the opportunity for pen pals to create continuous exchanges which developed over the course of the exchange. Finally, the use of both language and mathematical problems appropriate for the elementary school pen pal contributed to creation of a successful dialogue.

Factors Discouraging Development of Dialogue

Just as many factors combined to encourage the development of dialogue during a mathematical pen pal letter exchange, many factors worked to discourage the development of such a dialogue. A significant factor during the course of this project was disruption of the correspondence. Some pre-service teachers were absent for a week. Some informed me that they had chosen to end their participation in the project. Other letters never materialized and no explanation was provided. It was simpler to ensure that each of the elementary school student wrote a letter. Whatever the cause, silences effectively ended most of the discussions. The elementary school student quoted in the "Positive Attitude Toward Writing" section made it clear that the students did not appreciate lapses in communication from their pen pals. Occasionally letters from pre-service teachers arrived but did not include new problems. I encouraged recipients of such letters borrow problems from a classmate's letter and solve those as an attempt to patch this disruption in the flow of mathematical problems, solutions, and discussion between pen pals.

To a lesser degree, correspondence was disrupted when a pen pal referred to a problem in a past letter without restating the problem. The recipient of such a reference often could not recall the details of the original problem and was unable to reply. One elementary school student explained to his pen pal, "Unfortunately, I am unsure of the questions I gave you on the last set of questions."

Other times the dialogue appeared to have been intentionally disrupted by a student attempting to avoid a problem. One elementary school student, for example, did not answer one of the problems posed by his pen pal. She presented the problem again in her next problem, but he again did not respond to the challenge.

Distance and the concomitant delays in feedback between pen pals posed challenges to the establishment of a dialogue. The answer to a simple clarifying question would not be received for up to two weeks. This often forced both elementary school students and pre-service teachers to guess at what a pen pal meant. Distance made it more difficult for both parties to assess the motivations behind inactions as well. For example, a student may simply have overlooked a problem or chosen to skip it because it was too difficult.

The final factor which posed a significant challenge to establishing dialogue between pen pals was language. A student who began the year with minimal knowledge of English wrote letters in his first language, French. None of the participating pre-service teachers was proficient in French. I chose to reshape the task for this student, emphasizing understanding of word problems in English. Clearly, this student did not have access to the full value of discussion and dialogue in mathematics during this project. While working with this student in class we could respond to one another immediately and use various visual cues (such as acting out situations with manipulative materials or drawing pictures), advantages which were not available when communication was restricted to letters.

Dialogue was most significantly discouraged in this project by disruptions in correspondence between pen pals. These disturbances occurred for a variety of reasons. The absence of trust and respect which encouraged dialogue did in one case hinder dialogue. One pair of pen pals developed heated discussions of politics initiated by the pre-service teacher and stimulated by the Presidential election campaigns under way during the letter writing project. The university pen pal introduced himself as a "conservative Republican." The elementary school pen pal responded that he was a Democrat. His pen pal responded by writing,

> So, you are a Democrat? I am intrigued as to how you made the decision to be a Democrat. . . Why do you believe - as all Democrats do - in government control of people, high taxes, wasteful spending and thinking of human beings in terms of their race or gender instead of by the content of their character?

The elementary school student replied,

> Your criticism of democrats is shocking to me. I'm a democrat because they use money wisely (contrary to what you said) and republicans don't. Maybe they tax people more to use the money on useful things. Unlike republicans. You hint that democrats are racist and sexist. You're mixed up. . .

The heated political disagreements distracted both pen pals from the mathematical aspect of the project and kept them from developing the trust necessary to develop a dialogue in their letters.

Mathematical Questions and Content of Pen Pal Letters

For this research project I chose to focus on elementary school students' writing of mathematical questions for their pen pals. In my analysis I looked for patterns in the type and content of the problems they composed.

Computation and Word Problems

In the first letters my students wrote I asked them to include three of the four paragraphs from the letter-writing format (omitting the paragraph in which pen pals evaluate the responses they received to the last letter) and encouraged the students to "write three math problems for your pen pal." This occurred during the second week of school, therefore I was curious about their interpretation of this direction. The overwhelming interpretation was to write computational problems. Even the students who proved to be the most sophisticated mathematical thinkers in the class wrote computational problems. In response, I began encouraging students to write word problems by the time they wrote their second letters.

Large Numbers

Interesting patterns emerged from these computational problems. Most of the students wrote problems involving large numbers. Some addition and subtraction problems involved 23-digit numbers. It became clear that many of the students equated large numbers with difficulty in mathematics. One elementary school student gave his pen pal three problems, including 1,000,000 x 9,000,000. "Those are very hard to find out," he warned his pen pal. Another elementary school student gave his pen pal a multiplication problem involving a 27-digit number. His pen pal responded emphatically,

> Listen, kiddo, you've gotta stop hitting me with these HUGE numbers! The first question I always have when I read your big number problems is: 'Does [name] know how to write this/solve this?!'

The elementary school student acknowledged the impracticality of his problem, writing,

> I understand how you couldn't finish the problem. My calculator doesn't have 27 digits either!

Clearly, this student had not solved his own problem before sending it to his pen pal. More significantly, this student did not appear to realize that he might not be able to solve it himself. This student was not alone in presenting his pen pal with mathematical problems which he had not solved. The student giving his pre-service teacher pen pal a multiplication problem involving a 23-digit number and a 22-digit number had not solved his own problem. While students should be able to formulate questions in all subject areas which they would be unable to answer without assistance, recognition of this need for assistance is an important part of accurate self-evaluation. Students need to know what they do not know in order to focus their attention of learning those skills or finding missing information.

Unsolved Problems

Two groups of students writing unsolved problems emerged. Some of the students soon realized that without tedious effort they could not solve these problems themselves. One student wrote, "Next time I'll ask you an answerable question." These students, overall, wrote problems which they had solved or were confident they could solve. For these students their "problem posing could be considered an index of [their] problem solving" (Silver et al. 1996, 307).

Other students, however, never appeared to recognize that they were writing problems which could not be solved without tedious effort. The university pen pal receiving the multiplication problem with a 23-digit number and a 22- digit number responded by adding. She wrote,

> This would be the answer if I was adding. If it was multiplication the problem was <u>too</u> hard- it would take a long time to figure it out.

Her elementary school student pen pal responded that her solution was wrong because the problem was a division problem. He persisted in writing large number problems, however, and included a similarly large division problem in that letter. Not surprisingly, these students performed lower in mathematics throughout the year.

Building on Class Problems

The mathematical problems written by the elementary school students were clearly influenced by the classroom curriculum. Almost every problem and problem type appearing in the elementary school students' letters throughout the correspondence had an identifiable classroom antecedent, with the exception of some word problems. Most of the word problems, however, were taken directly from class or homework activities or were modifications of such activities. The million meal problem cited above, for example, and the number of kids who are required to circle the Earth's equator problem (modified by a student in the "Examples of Dialogue" section above) were both homework problems. Using familiar problems or problem formats has clear advantages for the students. They have solved these problems and often participated in a class discussion about the solution. This familiarity may have given many students greater confidence in presenting the problems to their pen pals and in later evaluating their responses.

Very few of the students, however, acknowledged that these were borrowed problems. Most simply presented the problems without comment. The three elementary school students who identified the sources of their problems were girls. It was not possible for me to identify patterns in the creation of problems by pre-service teachers, unfortunately, because I was not familiar with their classroom experiences. Only one pre-service teacher acknowledged a source for any of her problems. She explained that she had used one question in her mathematics methodology course.

Calculators

Calculators came up rarely in the correspondence. A few students commented on their limitations when dealing with large numbers. One student prohibited his pen pal from using a calculator to solve his problems.

Question Types

> Hans-Georg Gadamer, among others, has argued that it is the structure of *questioning* that gives a particular dialogue its tone and character (Burbules 1993, 96, emphasis in original).

Given the significance of questions in dialogue, I turned my attention to examining the types of questions appearing in pen pal correspondence. Although I could identify the connection between almost every elementary school student-composed problem and a class or homework activity, there was still a broad range of question types in the students' letters. Computational problems predominated, but other types of problems appeared in smaller numbers.

Problems with Multiple Answers

During the course of the correspondence, only three of the elementary school students presented their pen pals with problems which had multiple solutions. One of these problems, a multiple clue mystery number problem, was intended to have only one solution. Upon reading her pen pal's solution, the elementary school student remarked, "I seemed to have overlooked 7 on mystery number 2." Two elementary school students posed problems with multiple answers in response to similar problems written by their pre-service teacher pen pals. Clearly, neither group viewed problems with multiple answers as an essential question form. Problems with multiple correct answers, however, can reinforce the idea that solutions may not only be found through a variety of methods, but that more than one solution may exist. The preponderance of single-answer problems in the elementary school mathematics curriculum emphasizes the expectation that each problem will have only one correct answer which is reached through a standard algorithm. Despite my awareness of the value of problems with multiple solutions, I did not include them in the curriculum as frequently as I could.

Multiple Clue Mystery Number Problems

Three of the elementary school students gave their pre-service teacher pen pals multiple clue mystery number problems which they had attempted to solve in class. They had not found a solution, however, and chose to pass on the challenge to their pen pals. Another two students followed this mystery number format and created their own mystery number problems which involved multiple clues. One pre-service teacher composed a multiple clue problem to challenge her elementary school student pen pal into finding her age. Again, this question format was clearly not considered an essential question form for mathematics problems by either the elementary school students or the pre-service teachers.

Process Problems

One of the pre-service teachers wrote a process problem to her pen pal. Process problems challenge students to *explain* how they might solve a problem without actually solving it. The pre-service teacher asked her pen pal to explain how he would find the factors of a given number. Another pre-service teacher changed her pen pal's problem into a process problem, explaining how she would solve the problem if she knew how much a zillion really was. While process problems were clearly uncommon in the correspondence, most of the pen pals did ask their partners to explain how they arrived at their answers. Many pen pals, both elementary school students and pre-service teachers, included process explanations in at least some of their solutions. Some pre-service teachers reminded their pen pals to provide a complete explanation of a solution as a gentle redirection when the elementary school student had incorrectly solved a problem. For example, one elementary school student received a problem which required him to use percentages to solve it. He did not ask for assistance in class although the topic of percentages had not yet been covered in the curriculum. Instead he told his pen pal that contract A was the better deal for the movie star and moved on to another problem immediately. His pen pal responded,

> So, being able to show all your work and how you arrived at your answer is just as important as giving an answer. It's also important to show all your work because you'll be able to retrace your work and understand why you arrived at an answer. . . . I need to know not only why you chose A (contract A), but how you arrived at this answer.

This pre-service teacher has explained one of the strengths of using writing in mathematics. Reading student explanations of how they found solutions often illuminates their misunderstandings and areas in which they need assistance. The written dialogue between these two pen pals provided a venue for sharing insight and understanding. As the written conversation bounced between pen pals, they had opportunities to explain themselves and ask questions to help them better understand one another.

Although process problems were uncommon in the correspondence, process explanations were much more common and helped pen pals better evaluate one another's responses and provide assistance when needed. Weekly homework assignments for the elementary school students required them to write detailed explanations of the process used to solve a particular mathematical problem.

Catch and Correct Error Problems

One elementary school student wrote a problem for his pen pal which challenged her to look at a problem and solution in order to find and correct an error. Every pen pal, however, potentially posed such a problem by simply solving a mathematical problem. When evaluating these solutions, each pen pal needed to determine if an error was present in the solution. If so, the student needed to explain the error to the pen pal. The problem format of examining a problem and searching for an error was present in the word problems which the elementary school students solved each week for homework.

The elementary school students often enjoyed catching and correcting errors of their older pen pals. One elementary school student wrote, "I am supposed to evaluate your answers but I will only do one and it is the Christmas one. YOU ARE WRONG!" He then explained the complete solution to his pen pal and identified the error she made in calculation. Student interest in correcting errors was not limited to mathematics. One elementary school student informed her pen pal that she had changed tenses in the middle of her letter.

Missing Information Problems

At least three elementary school students and three pre-service teachers wrote problems for their pen pals which had important information missing. Interestingly, most of the pen pals appeared uncomfortable with this format and ended up supplying the missing information for the partner. One elementary school student, for example, wrote,

> If 1,000,000 kids stood on each other's heads how tall would they be? (Hint: They are 4 feet tall!!!!)

A pre-service teacher wrote,

> How many gallons of water do you drink in one year? (Let's assume you drink 1/2 gallon a day.)

Perhaps they were concerned that the pen pal would simply reply that information was missing instead of providing the information and solving the problem. The missing information format was used in some class homework assignments in which the missing information aspect of the problem was clearly identified.

Looking for a Pattern Problems

Most of the problems which challenged a pen pal to find a pattern were written by the pre-service teachers. Two university students wrote number series problems for their pen pals, challenging them to extend the series. Another two pre-service teachers wrote problems presenting a series of multiplication problems and challenging the elementary school pen pal to find the pattern. Only one pre-service teacher asked her pen pal to examine the answer to a single problem and find a pattern in that answer. Two of the elementary school students responded to their pattern problems with pattern problems of their own. One made the number series problem more difficult by interspersing the blanks within the sequence instead of putting them all at the end of the series.

Writing Original Word Problems

As mentioned above, the elementary school students did not spontaneously write word problems for their pen pals. With encouragement, however, they began to include word problems. Many of the problems were taken directly from, or based on, weekly word problems given in class. Some elementary school students, however, wrote original word problems. Their skill at writing the word problems seemed to correlate highly with their overall performance in mathematics. Some of the students wrote original word problems which required multiple steps and multiple operations in order to be solved. Elementary school students, for example, wrote the following:

> There are fifty spiders in a box. 29 spiders are missing one leg and three are missing two. How many legs are there in all?

> If Ben and Jerry sold 3,406,093 ice cream cones a year and sold 6,349,665 cups a year, how many ice cream cones and cups together would they sell in 23 years?

> If you were born in 1933 and you lived until 1968 and you started knitting at 3 years old and knitted 8 scarves a day, how many scarves would you knit in your life?

> You are a miner in the California gold fields. Your claim is very rich. You get $50 a day. A decent meal is about $10. You eat 3 a day. You want to build a roomy cabin. It will cost $10,000. The theater is open once a week, and you go every time. It is $10. How long do you have to save to build your cabin?

These elementary school students and the others who wrote multi-step word problems which were comprehensible and solvable were successful mathematics students throughout the school year. Other students never ventured to create completely original problems. They stayed with the familiar problems used in class or modified them slightly, and they were more likely to be average or lower than average mathematics students.

The final group consisted of elementary school students who attempted to write original or modified word problems but often wrote simple one step problems or created incomprehensible or unsolvable problems. Examples of such problems include the following:

> Ice cream is one of my favoraite [sic] foods. If I had ice cream 2 times on Mon., 1 time on Tues., 6 times on Wed, and 1000 times on Fri., how many times would I eat ice cream?

> If you wach [sic] 3 hours of T.V. in January and you wached [sic] 1 in February in 1995, how many minutes did you wach [sic]? (You need to use a 1995 calendar.)

> How many pennies and how tall would they be in 1,000,000?

> With this number here (4,973) times that number until you get NEAR!! one million. Show all the times you tried.

> There are three soccer fields. There are 210 kids. The AYSO soccer league needs to separate each team for a soccer game. They only have till 7:30 to 6:00. How long will it take to separate each team?

These were often the same students who wrote unsolvable computational problems for their pen pals. Although some of these students were better able to solve problems given to them, overall, the students who had great trouble writing word problems were less successful in mathematics.

Riddles

Nonmathematical riddles appeared in the letters written by three of the elementary school students. Two of the students had worked together in class writing their responses and talked about including riddles. They included a riddle in almost all of the letters they wrote and enjoyed posing riddles to each other all year. The third student worked independently and only included a riddle in one letter. The pre-service teachers responded to the riddles, but did not pose any.

Mathematical Content

Just as the question formats used by the elementary school students reflected the question types used at school, the mathematical content of the elementary school students' pen pal letters clearly reflected the content of mathematics instruction in class. The focus at the beginning of the year was place value and special numbers. This emphasis on reading large numbers was quite clear in the use of large numbers in the computational problems written by students.

Students also frequently asked their pen pals to give the value of a specific digit in a number or to write out the name of a large number in words. These were activities which were addressed in class. The special numbers addressed included factors, multiples, prime numbers, composite numbers, square numbers, triangular numbers, and palindromes. These concepts were each included at least once in a student letter. Again the students often gave their university pen pals the same kind of problems which they had solved in class. They asked their pre-service teacher pen pals to find all of the factors of a number, for example, or find all of the prime numbers in a given range. The elementary school students often included definitions of these terms in their letters.

Other mathematical topics appeared significantly less frequently in pen pal letters written by the elementary school students. The few measurement problems asked by elementary school students appeared to have evolved from two questions addressed in class. The first asked about the number of students it would take to stretch around the Earth's equator if they all held hands. The second, appearing in a book read to the class, asked how high a stack of 1,000,000 pennies would reach. Two pre-service teachers responded to questions from their elementary school pen pals with clarifying questions or extension problems which involved measurement. One pre-service teacher explained that she was preparing a unit about teaching measurement to fifth and sixth graders and asked her pen pal what units she would use to measure a variety of items.

The topics of time, probability, fractions, decimals, and algebra also appeared rarely in the letters. Time appeared in only two original word problems written by the elementary school students. Both were included in the "Writing Original Word Problems" section above in Chapter Four. One pen pal responded with a time-oriented problem.

Probability similarly appeared in only one letter. A pre-service teacher asked his pen pal how many socks he would need to pull from a drawer containing three colors of socks to be sure to have a pair. Fractions did not appear in any of the problems written by the elementary school students.

One pre-service teacher wrote a problem which required a fraction for the solution. His pen pal, however, did not answer the question.

While decimals were used in the context of money, they were similarly uncommon in other contexts. Only two elementary school students wrote problems involving decimals. One included the decimal 4.5 in an equation while the other student's final solution to her own problem was expressed as a decimal which she correctly interpreted in the context.

Algebra was only raised as a formal subject by one student although other elementary school students did include simple algebraic equations in their letters. I found it particularly interesting that the topics of decimals (outside of money) and algebra were raised only by elementary school students. In both cases, the students initiating these topics were above average mathematics students.

While the topics of decimals and algebra were only initiated by the elementary school students, the topic of percentages and negative numbers appeared primarily in letters written by pre-service teachers. While three pre-service teachers included percentages in problems they wrote, only one elementary school student did so. When the student received a percentage problem in return and needed to solve it, he responded, "Ahhhhhh! This was hard! I think I got it though!" Another elementary school student who received a percentage problem solved it successfully. The other two who received percentage problems either did not answer this part of the problem or clearly guessed at a solution. Negative numbers were thoroughly introduced and practiced in the context of addition in the correspondence between one pair of pen pals. One elementary school student used a negative number. She included a negative number in the same equation which included the decimal 4.5.

Clearly, the mathematics in the elementary school students' pen pal letters reflected the content addressed in class. Many of the topics which arose rarely in the correspondence had not yet been addressed for the year. Some topics, specifically non-money decimals and percentages, had not been addressed in previous grades.

Teaching Strategies

Many of the pen pal letters provided clear examples of one pen pal teaching the other. While the pre-service teacher more often helped an elementary school pen pal, there were examples of the reverse. The second dialogue in the "Examples of Dialogue" section above in Chapter Four included one such example. A number of strategies were used by pen pals to teach one another. Pen pals would provide direct and indirect assistance to one another for a problem, give positive feedback, and model their thinking about a concept or process. Pen pals rarely used pictures or explanation of alternative solution approaches as strategies when writing to each other.

Providing Assistance

As detailed above in the "Factors Encouraging Development of Dialogue" section in this chapter, many pen pals felt comfortable asking for help. Pen pals would usually respond to these requests.

At times the dialogue appeared to have been intentionally disrupted by a student attempting to avoid a problem. Some pre-service teachers interpreted this behavior as an indirect request for assistance. One pre-service teacher, for example, chose to respond by offering help indirectly. She responded to his letter, saying that the problems he had given her were very difficult. She ended her letter with an easier word problem for him followed immediately by the sentence, "Let me know if this was easy or difficult." Her elementary school student pen pal responded that the problem was easy, adding that his first grade brother could probably solve it. He also asked her in return to tell him if the problem he gave her was difficult or easy. In this case the dialogue was kept on track as the pen pals reached out to one another and sought to adjust problems to match their partner's abilities. This effort to keep both parties successfully involved in the discourse reflects the pen pals' commitment to the dialogue as well as their respect and concern for one another.

Clarifying Directions

Some pen pals made an effort to clarify directions for problems they posed. One pair of pen pals used examples to help with this clarification. The pre-service teacher first wrote a two part problem. The second part asked what the highest place value was in the answer to the first part. He then wrote, "Example: If your answer was $1,328.57 the highest place value would be the 1 and that has a place value of 1 thousand." His elementary school pen pal provided a clarification only after his pen pal had misunderstood his directions for a problem. He explained, "What I mean on the third problem (what do the digits add up do? 673)? The answer is 6 + 7 + 3 = 16. I hope you see how I did it." Although pen pals rarely clarified directions for one another, especially before a misunderstanding was apparent, they did frequently provide each other with positive feedback.

Responding to Pen Pals

The "Encouragement" section at the beginning of this chapter details examples of the positive comments which both elementary school students and pre-service teachers bestowed upon their pen pals. Sometimes, however, pen pals needed to respond to errors made by their partners. Often the partner would simply explain the correct answer. Occasionally, pre-service teachers would address an error by taking responsibility for it and gently directing the elementary school student to attempt the problem again. One student, for example, received a question about how big a box she would need for one million pencils. She estimated that the box would be one foot deep and one foot wide. Her pre-service teacher pen pal responded with a compliment about her use of estimation. "I forgot to mention that they would be new pencils," wrote her pen pal. She provide her pen pal with the dimensions of a new pencil and encouraged her to start with a simpler problem, determining how many new pencils would fit on the bottom of a one foot cube. One pen pal similarly emphasized her elementary school pen pal's correct answers before encouraging her to reconsider her incorrect answer.

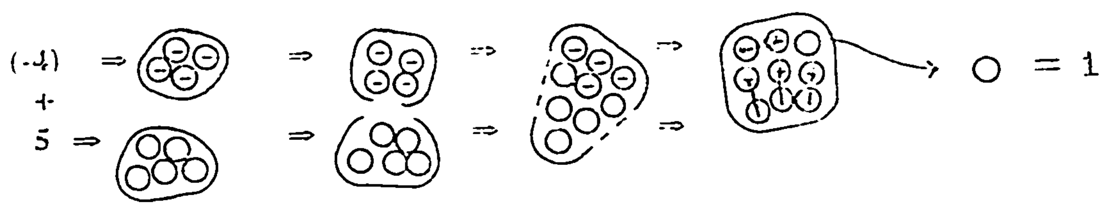
> There are 4 dogs, 3 cats, 5 chickens, and 1 horse. You said there would be 13 tails and 62 legs. You were right! [smiling face] Good job!! But, what about the ears; you said there would only be 13. Do you still think that is right, or would you like to change your answer?

In other situations, such as the one detailed in the "Factors Encouraging the Development of Dialogue" section above in Chapter Four, pen pals attempted to clarify their problems and provide their pen pal with more detailed questions to guide them.

Modeling to Teach a Concept

The strategy used by pen pals in their letters which appeared the most like direct teaching was the use of modeling by "thinking out loud." Pre-service teachers used this strategy to explain concepts which were often applied in the problems that followed. Ten of the pre-service teachers, almost half of the group participating, used this strategy in their letters. One pre-service teacher, for example, introduced his pen pal to negative numbers, explaining,

> They can also be written as: (-1) (-2) (-3) (-4) (-5) (-6) (-7) (-8) (-9) (-10) (-11). . . . The parenthesis is to show that the negative sign is attached to the number and that it is <u>not</u> an OPERATION.
>
> Negative numbers act differently when they are grouped together with counting numbers.



> They annihilate each other.
>
> Remember, to me, bringing two groups of objects together and giving them a single number is addition. In the grouping above, instead of becoming a bigger number like counting numbers, the two types obliterated each other until only one type remained and there are now less.

His elementary school pen pal was able to apply this lesson successfully to the practice sheet of addition problems involving negative numbers which he received with the letter.

Another university student used a problem she had received as the starting place for her lesson about the order of operations and use of parentheses when writing equations. She wrote,

> I don't know if you have learned this yet, but when you don't include parentheses, there are rules on what operations you do first. . . . So if you wanted everything done in a certain order, you would have to add parentheses. Since you didn't have parentheses, I did multiplication and division first, then addition and subtraction. I'll show you on a separate piece of paper.

Her elementary school pen pal replied, "The sheet you answered my second problem on was pretty confusing, but it looks interesting." The pre-service teacher responded, continuing the dialogue,

> I was hoping that my answer to your problem wouldn't be too confusing. I was just trying to show you that it is important to know in what order you do multiplication/division or addition/subtraction in an equation. The main thing you should learn from my explanation is that you should include parentheses; they tell you what to do first. If you don't include parenthesis, then you have to multiply or divide first. Example: 1 + 2 x 3 without parenthesis is = 7. Because for an equation, you do the multiplication first (2 x 3) is 6 plus the one is 7. Now if you try it in your calculator you'll get 9. Why? Because the calculator has no idea what numbers you'll put in for the entire equation. . . . It adds the 1 + 2 to get 3 then x 3 = 9. So if you did want the 1 + 2 added first then multiplied, then you should state the equation as such: (1 + 2) x 3 = 9. Try these and remember to do the numbers in the parentheses first, then multiplication, then addition. You should get four different answers.
>
> (1 + 2) x 5 + 1=   1 + (2 x 5) + 1=   1 + 2 x (5 + 1)=   (1 + 2) x (5 + 1)=

After the second explanation, the elementary school student wrote, "I really learned something when I read your letter. Sorry I didn't put in parenthesis (sic) because I didn't know you had to have them."

Elementary school students did not use the "thinking out loud" strategy to teach a new concept, but they did use it to explain the process used to find the solution to a problem. This topic was detailed in the "Process Problems" section in Chapter Four.

Modeling to Teach a Process

Pre-service teachers also used the modeling strategy of "thinking aloud" to teach their pen pals a process. Pen pals frequently used this strategy when explaining the process used to solve a problem from an elementary school student. For example, one pre-service teacher received the problem 1,000,000 x 9,000,000. He included his solution in his response and wrote,

> Your [sic] probably wondering how I did it? Maybe not!! I added the zeros. Any time you multiply by 10, you must add zeros. For example, 100 x 10 is 1,000. Now you try. What is 100 x 100?

Another pre-service teacher explained the process she used to find out how many fifties were in a million. She wrote,

- There are 10 100's in 1,000
- So, there are 20 50's in 1,000
- There are 100 1,000's in 100,000
- 20 (50's) times 100 (1,000's) equals 2,000 50's in 100,000
- There are 20 100,000's in 1,000,000
- 2,000 (50's) times 10 (1000,000's) equals 20,000 50's in 1,000,000

The elementary school students did use these "thinking aloud" strategies, but less frequently and with less detailed explanations than their pre-service teacher pen pals.

Rarely Used Strategies

One strategy that proved quite rare within both groups was the use of pictures. The pre-service teacher cited above teaching about negative numbers made pictures integral to his explanation. When used by the other two pre-service teachers, the pictures supported, but did not carry, the explanation. Two elementary school students encouraged their pen pals to use pictures, but only one used any in her letters. The pictures that she did draw, however, were illustrations of the context, but not the mathematics, of the problem. Pictures were clearly not used by many participants as a teaching strategy.

Unfortunately, pre-service teachers often missed opportunities to link problems to the real experiences of their elementary school pen pals. The university student mentioned above, for example, did not extend the learning opportunity by challenging his pen pal to examine and explain the pattern of "adding zeroes" which emerges when multiplying multiples of ten. The preponderance of problems with single answers also reinforced the notion that mathematics is composed of discrete problems which are best solved through common computational methods. The pen pal letter exchanges rarely encouraged students to consider open-ended questions or probe for patterns beneath answers.

Using the Pen Pal Correspondence in the Classroom

Reading the mathematical pen pal letters and working with the students in class as they wrote their letters helped me recognize some student understandings and misunderstandings about mathematics. A letter often brought to my attention specific information about an individual student's understanding of an individual concept. Using the letters I was able to quickly identify students who were having trouble with basic computation skills, for example. As part of my classroom curriculum, I regularly identify small focus groups of students to work with a parent volunteer on a specific mathematical concept or skill. The pen pal letters became one means of identifying students in need of individual or small group assistance.

Patterns of Student Achievement

Broader patterns of understanding and misunderstanding across the class emerged as I read the complete collection of letters. Extensive use of classroom and homework problems in the letters made it clear to me that most of the students were able to identify the topic we were studying in class and provide examples of relevant practice problems. While I believe that the preponderance of large numbers in the pen pals letters reflected the classroom focus on place value and large numbers, I also think that fifth grade students consider large numbers challenging in general. Clearly, the possibility of making a minor calculation error is greater with larger numbers.

A significant finding to emerge from examination of the complete body of data, however, was the high degree of correlation between overall mathematics achievement throughout the year and the presence of specific characteristics in the pen pal letters. This analysis enabled me to formulate profiles of the most and least mathematically successful students of the class based on their letters.

My definition of mathematical success includes the combination of sixth grade mathematics class placement for the students and grades on class mathematics units. Sixth grade mathematics class placement in my school is based on the combination of teacher assessment of student study skills and speed of concept mastery in mathematics as well as standardized test scores in mathematics and scores on school-designed tests of both mathematical computation skills and conceptual understanding. The criteria for placement into the accelerated section of sixth grade math strongly favor students with strong computational skills and quick insights into mathematical content. Grades on class mathematics units, however, were more holistic, taking into account growth in conceptual understanding shown through writing and problem solving as well as application of concepts to situations. Despite the differences between the two assessments of mathematical success, there was a link between the two groups. The most mathematically successful students received grades of A or B on class units and achieved placement into the higher mathematics class for sixth grade. The least mathematically successful students received low B and C grades on math units, received assistance often in the small focus groups, and were not placed into the higher sixth grade mathematics class. These two categories each represent approximately one quarter of the class. The remaining half of the students fell somewhere in between and cannot be described clearly by either profile.

Most Mathematically Successful Elementary School Students

The most mathematically successful elementary school students were also skilled letter writers. They were able to write explanations easily and quickly and included relevant details in their explanations. These students responded easily to a range of topics and question types, individually researching needed information at times. These students asked for assistance or clarification at times, but were generally able to take responsibility for completing their letters independently.

This group contained students who responded carefully to each question posed by their pen pals and wrote comprehensible word problems which often required multiple steps to find a solution. Their letters were among the longest written by the elementary school students. These students generally evaluated the accuracy of their pen pals' responses, often making specific comments about their solutions, and usually solved their own problems before sending them to pen pals. Some of these students introduced mathematical topics which had not been addressed in the school curriculum before the letters were written. Clearly, these students had strong language and academic skills. The mathematical content of their letters, however, reflected solid understanding of the concepts addressed as well. These students generally expressed an enjoyment of mathematics and realized that they were successful in this area.

Least Mathematically Successful Elementary School Students

The least mathematically successful elementary school students had a wider range of writing skills. They usually wrote terse responses to problems, often omitting any explanation of how they had arrived at a solution, were more likely to avoid a question by not answering it, and were more less likely to ask for assistance or clarification than their classmates. When I looked over their work in class, however, I would often find an area in which they needed assistance. These students were generally able to take responsibility for completing their letters, but wrote shorter letters.

This group included the students who seemed most convinced that large numbers made a problem difficult, with two of them stating this belief directly in letters. Many persisted in using large number computational problems throughout the pen pal project despite complaints from their pen pals. In class, the initial difficulty of reading large numbers was discussed and many class activities involved investigations of large numbers to help the students develop an understanding of the magnitude of one million in particular. Word problems were rarely written by these less mathematically successful students when considered as a group, although two of these students did regularly attempt to write word problems. These students wrote only one word problem per letter. These problems, whether original or based on class problems, were often confusing or incomplete. They rarely required multiple steps to find a solution. Most often they used the operations of addition and subtraction, and often wrote problems which they clearly had not solved. These students had difficulty connecting mathematics to real life situations. Many of the computational problems were completely impractical or required an understanding of decimals to interpret the answer. These students appeared not to evaluate the accuracy of their pen pals' responses, simply writing that the pen pals had been right or wrong. Clearly, this group of students had internalized the idea that mathematics is all about computational problems with clear right and wrong answers. This was, of course, a logical understanding for them to develop given the tradition of teaching mathematics in the United States. Many, but not all, of these students also had lower language and academic skills. The mathematical content of their letters, however, reflected incomplete understanding of many of the specific mathematical concepts addressed. Interestingly, there did not appear to be a strong connection between this lower level of mathematical performance and affinity for the subject of mathematics. At least one of these students considered mathematics his strongest subject.

Letters as a Reflection of Classroom Instruction

Student writing about mathematics provided an interesting mirror reflecting my teaching of mathematics. Seeing the problems I had presented retold or altered by students and reading their descriptions of classroom activities made it clear that they generally were paying attention, often recalling problems in surprising detail. The students recognized the topics taught and showed a general understanding of the relevant concepts and skills. Most of the students were able to recognize and solve the application of a concept in word problems. Fewer students, however, were able to make such an application themselves. This ability to make applications was most clearly illustrated by the students' creation of word problems.

Reflections of the Question Types I Pose

> [It is clear that] the central influences that develop the capacity for questioning in students are the sorts of questions teachers model in interchange with them; the manner in which teachers respond to the questions they are asked; and the sorts of questions teachers encourage students to ask themselves and one another (Burbules 1993, 100-1).

Examination of the types of questions written by my students directed my attention to the types of questions I posed to them. The types of problems they wrote roughly reflected the types of problems I presented to them. If all of the problems I address in class and assign for homework were considered individually, probably three quarters of the problems I assign are computational. The bulk of these problems are skills practice problems assigned as class warm-up or homework from the mathematics textbook.

Distortions in the Reflection of the Problems I Posed

The types of word problems written by my students appeared to be significantly less reflective of the word problems I gave to them. In addition to the computational problems, I assigned an assortment of word problems each week. Homework regularly included a set of five word problems from a published curriculum and a longer problem of the week which I either modified or created to accompany the class curriculum. The set of word problems were structured to include one simple computational problem in disguise as a word problem each week. These were generally one or two step problems which can be solved with whole numbers and use of the basic operations. Students most often wrote this type of word problem although they were not the most common type of word problem assigned during the year.

The other problems included in the weekly word problem set were an assortment of problems requiring greater application of logic and other mathematical skills. These word problems included problems with multiple answers, process problems, problems requiring identification and correction of errors, missing information problems, opportunities to write of original word problems, and problems requiring the creation of organized lists. The longer problem of the week assignment emphasized a variety of problem solving strategies and writing a detailed explanation of the solution process followed by the student. The students and their university pen pals wrote few problems of these types, instead writing mostly one or two step computational problems. My students also wrote problems types which I did not. I did not assign incomprehensible word problems or write problems which could not be solved. My students, however, wrote both of these types of word problems. Clearly, some students were struggling to create their own word problems, but they were using classroom models to assist them in this endeavor. It is, as Gadamer observed, "more difficult to ask questions than to answer them" (1982, 326, cited in Burbules 1993, 101).

Shaping the Curriculum

While these letters were written at the beginning of the year and may be more reflective of the students past experiences with word problems than their exposure to word problems in my classroom alone, I consider this feedback important to my instruction. Considering both the characteristics of the least mathematically successful students and the types of word problems written by the class as a whole, I think that my instruction should place greater emphasis on student creation of word problems. The pen pal letters represented the most significant opportunity for students to compose word problems during the last year. There was not a formalized classroom or homework routine which ensured practice with this skill throughout the school year. Yet writing word problems requires students to apply mathematical concepts and skills, forging connections between these abstract notions and real life situations. This appears to be a particular weakness for my least successful students.

Another curricular implication of this analysis appears in the area of content topics. The topics appearing in student letters clearly reflected those studied in class. Measurement, however, is present among most mathematical topics but appeared rarely in student word problems. Students occasionally omitted units from problems as well. As a result of this understanding, I would like to raise my awareness of the measurement aspects of the topics I teach as well as monitor student use of measurement terminology and concepts in their word problems.

Taken together, these two measures will increase the emphasis on word problems in my mathematics instruction. Not only will I continue to challenge my students to solve a variety of word problems, but I will also require them to create their own word problems on a regular basis and monitor their use of measurement in the word problems. An important component of including word problems in the curriculum will be evaluation of these student-written word problems. This was another area of weakness for the least mathematically successful students in my room. Students are unable to monitor their own performance if they do not learn to carefully evaluate work, especially their own. By examining student-written word problems in class, often using them as the content for instruction or practice, I hope to raise student awareness of the different types of word problems as well as improve the level of student skill at creating word problems which connect mathematical concepts to real life situations.

Connecting the Data to the Research

After wading through the stacks of pen pal letters, I struggled to connect the letters to my research. I finally realized that I had read about learning, reading a largely philosophical and theoretical body of work about how students learn concepts and skills. Believing that learning is an evolutionary process with each student's body of knowledge and skills growing gradually, I tried approaching the letters as portfolios, reading the collection of correspondence between one of my students and a pre-service teacher. I thought this might help me find confirming or discomfirming evidence about how my students were learning about mathematics. Instead, I found that each collected correspondence showed me at best how an individual student was thinking about a mathematical concept at that moment in time. In some cases, the letters were too disjointed to shed light on the student's thinking. Each letter proved to be a snapshot, a window into a specific student's thoughts at one point in time. This is the point of a portfolio in which many of these limited glimpses into a student's thinking helps to create a more complete view of the student's learning.

Learning From My Research

As I read about the conditions which encourage or support teacher learning and pedagogical change, I considered my own situation (Maher and Alston 1990). Opportunities for mathematical explorations with peers were provided through my participation in the Santa Clara Valley Mathematics Project, a California mathematics curriculum institute, as well as within courses at San Jose State University. Opportunities to discuss the philosophical basis for and pedagogical implications of change were available to me as I worked closely with my grade level teaching partner for three years, often discussing these issues together. My involvement in the Program Quality Review (PQR) program as both a member of the staff of a school undergoing a self-evaluation and improvement process in the area of mathematics and as a consultant for another school going through the same process provided further opportunities to discuss the philosophy behind changes in mathematics pedagogy. The in-service training provided by my school district to help teachers implement the *MathLand* program (Creative Publications 1994) provided a very clear context for discussions of how constructivist ideas could influence instruction.

In contrast, I had fewer opportunities to learn about how children interpret ideas in mathematics, what strategies children use in mathematics, and how to interpret errors made by students. Much of my understanding in this area grew from my experience in the classroom. I simply found these questions intriguing and watched for patterns and analyzed student errors by trying to understand why students had formulated their solutions as they had. This personal learning was strengthened, however, by discussions with other teachers, my reading of reform documents during the PQR process and at the curriculum institute, and my use of mathematics replacement materials. The background reading completed for this research considerably improved my understanding of children's mathematical thinking. These conversations provided a forum for me to share my observations and questions with others who shared my interest. These dialogues were conversations emphasizing understanding or inquiries with a specific question to be answered (Burbules 1993). The articles and books often presented me with new notions to consider or new frames for viewing the same situation.

Challenges of Mathematical Vocabulary

Mathematical writing has limits inherent to the use of language in general and writing specifically. Languages have mathematical vocabulary, words created to represent concepts. These words, however, both give us power to explain mathematical ideas and can also limit our explanation. Language reflects a way of organizing and classifying the world and our experiences in it (Brown 1996). Writing about mathematics is similarly more challenging for some students. These students often produce mathematical writing which fails to reflect the depth of their understanding (Miller 1992b). Writing may also be significantly more challenging than the use of algorithms for students just acquiring English. Those students who have acquired some English, however, may prefer writing to speaking because of the opportunity it affords them for revision of their writing and the use of a dictionary to assist in composing a response.

Related Findings

L. Diane Miller and David England (1989) found that students wrote more in response to prompts if they were directed to address their response to someone such as a friend, classmate, or younger student. Additionally, they found that simpler, more direct questions elicited better responses from the students. "Longer, more detailed prompts yielded *less* writing, despite their intent to generate *more*" (310, emphasis in original). More complex prompts asked students to address more than one question and frustrated struggling students (Miller and England 1989). The number of questions may also have reduced the time available to the students to address each individual question, reducing the quality of the responses. Finally, student writing was found to improve over time as the students developed familiarity with the task (Miller and England 1989).

Summary

This examination of the mathematical pen pal correspondence between elementary school students and pre-service teachers sheds light on both the social and mathematical dimensions of the letters. Most writers readily established a positive social environment in the letters, making dialogue between the elementary school students and pre-service teachers possible. Mathematically, the influence of the classroom mathematics instruction was clear. The problems posed in class were reflected in the question types and mathematical content in the problems written by the students. The implications of these findings are examined in detail in Chapter Five.

CHAPTER FIVE

IMPLICATIONS, CONCLUSIONS, AND SUGGESTIONS FOR FURTHER WORK

Replication and Classroom Application

Using children's writing in mathematics may prove to be a powerful tool, helping many teachers to better understand their students' thinking and understanding about topics and empowering students by involving them in the creation of the curriculum. To facilitate the wider use of writing as a teaching strategy for teachers of mathematics, other teachers and researchers may wish to use this project as a springboard to their own projects.

Replicating the Mathematical Pen Pal Project

Teachers wishing to implement a mathematical pen pal project as a classroom action research project should consider selecting a focus for their project. The data available from a pen pal correspondence can be overwhelming. A predetermined focus may help a teacher-researcher analyze the data during the project. For example, a teacher may wish to focus on the effect of the correspondence on students' problem-posing skills, their abilities to evaluate mathematical responses by others, possible gender differences in the quality of written explanations, changes in student skills or understanding in a specific area over time, or the effect of the correspondence on student attitudes toward mathematics instruction. Teacher-researchers should expect that the mathematical content of the student letters will reflect the topics being addressed in class at that time. This focused approach may be most appropriate for classroom teachers trying to use writing to learn more about student understanding in order to immediately (and in an on-going fashion) tailor instruction to student needs. Examining all of the data later, searching for patterns will, certainly yield unexpected findings, but a classroom teacher may have a more narrow purpose in implementing the project. The data could be examined in both manners, of course, if the researcher so chooses.

Some modifications will make it easier for teacher-researchers to foster continuing, dialogic discussions between pen pals. First, have students keep copies of letters so that they can remember what they wrote in their last note. This inability to recall the specifics of a past letter often hampered communication between pen pals. Second, encourage the older pen pals to reiterate problems when they solve (or ask questions about) them in the reply. Pen pals should not assume that the other party remembers the details of the problems they wrote. A third change will help encourage the younger pen pals to solve their own mathematical problems, reducing the tendency to write impractical or unsolvable problems. Students should make answer sheets for the problems they write and keep this answer sheet with the copy of the letter for reference when the reply is received. These practical adjustments should improve the quality of the correspondence between mathematical pen pals.

Expanding the Use of Writing in Mathematics Instruction

Exploration is just beginning in the combination of writing and mathematics. Very little research has been published on this and much of what is available focuses more narrowly on the understanding of word problems in mathematics. At this point, the use of writing in mathematics as an assessment tool, along with other measures, has the potential to improve on-going classroom instruction by providing teachers with a more complete picture of what their students really understand.

Introducing the Pen Pal Model to Classroom Teachers

The publication of further work, particularly shorter articles in professional publications read by many classroom teachers who may not be currently affiliated with a university, detailing the experiences of teachers using writing as part of their mathematics instruction may encourage many teachers to introduce some writing into their mathematics curriculum. I have found the vignette style of writing about classroom experiences to be the most helpful when I try to imagine implementing a practice in my classroom which I have not seen in another classroom. I believe that other teachers will find it easier to introduce some writing into their math instruction is they have access to qualitative examples of how this practice may work written by real teachers about their own classroom experiences.

Simplifying Writing in Mathematics for Classroom Application

Establishing a pen pal letter exchange may require better access to university faculty, or more effort, than a classroom teacher wishes to expend, especially when first incorporating writing into the mathematics curriculum. The letter writing project could be simplified, however, and still provide useful information to the classroom teacher. Students could write letters to older or younger students from local schools using the same format. The students could write letters to the teacher with the understanding that they would be answered as a class instead of individually. Students could correspond via electronic mail instead of pencil and paper. The question-posing portion of the letter writing could be incorporated into tests or other assessment measures already used in class. For example, a division test may include the assignment to write and solve a word problem whose solution requires the use of division. Finally, elementary school students could write non-letter responses to focused assessment questions. Simply asking students to write an explanation for why they solved (or posed) a problem as they did can shed significant light onto their mathematical thinking and understanding. Certainly there are many other modification options available to teachers who are interested in incorporating writing into their mathematics instruction.

Enhancing School-University Collaboration

Collaboration between the university and elementary school was essential to the implementation of this project. Without support from university faculty, pre-service teacher participation would not have been possible. While the elementary school students could have written mathematical pen pal letters to other students and received the same individualized attention and mathematical modeling, the participation of the pre-service teachers offers other benefits to both participating groups. The elementary school students have the opportunity to interact with adults with a demonstrated interest in and commitment to elementary education. The pre-service teachers are afforded the opportunity to establish a continuing dialogue with an elementary school student. This communications allows the pre-service teacher to connect and apply their instruction in a mathematics methodology course to a real student. Other teachers could use this model of mutually beneficial collaboration to action research projects they undertake.

Including Pre-Service Teachers in the Pen Pal Project

> After all, most of us are products of elementary and secondary school classrooms in which the teachers told us what we needed to know or do and we listened to and did what they told us to do (Vacc 1993, 88).

Researchers working with pre-service teachers have found that pre-service teachers tend to teach in their student teaching more as they were taught as young students, often not applying the strategies or philosophies they were taught in a teacher education program (Eisenhart et al. 1993, Schiebelhut 1994). One study attempted to examine the causes for this phenomenon and found that the school environments in which the pre-service teachers student taught and the pre-service teachers' reactions to the often conflicting pressures from the school administration, master teacher, and university about the relative importance of conceptual and procedural knowledge combined to create this tendency for pre-service teachers to teach in a more traditional fashion (Eisenhart et al. 1993).

Pre-service teachers represent the future of teaching. Margaret Eisenhart and her colleagues(1993) claim,

> Given the importance of the routines and patterns established during a teacher's beginning years (Feiman-Nemser, 1983), it seems important to ask what must happen to increase the likelihood that student teachers will have the opportunity to teach and learn to teach for conceptual knowledge in accord with the mathematics education reform agenda (37).

One possibility is a closer collaboration between university and public school personnel to help develop opportunities for pre-service teachers to work in classrooms in which mathematics is taught for understanding. Writing mathematical pen pal letters can provide a class of pre-service teachers with opportunities to practice and improve in their use of questioning to help understand student thinking before they enter the classroom as student teachers.

<u>Problem Solving Interviews of Individual Students</u>

Harriett Bebout (1994) introduced a class of pre-service teachers to Polya's four steps of problem-solving: understanding, representing, solving, and checking. Each pre-service teacher then individually interviewed a student who had been identified as a "low" student in mathematics. The interview was structured around a problem solving activity so that the pre-service teachers could watch for the use of the four problem solving steps. Many of the pre-service teachers found that their subjects could solve very challenging problems when given opportunities and materials.

There are several parallels between this model of a pre-service teacher interacting one-on-one with an elementary school student and the pen pal letter exchanges between elementary school students and a class of pre-service teachers. Both situations provide pre-service teachers with opportunities to learn more about how elementary school students really think and respond in a mathematical context. They provide pre-service teachers with an opportunity to apply or examine some of the information they are learning in teacher education programs with real students. Finally, elementary school students in both instances have the unusual opportunity of an extended period of individual attention from an adult. This model of problem-solving interviews could be used in tandem with a letter-writing project to provide pre-service teachers (or practicing teachers) with more information about student understanding of mathematical topics.

Studying the Pre-Service Teachers

In addition to providing access to a valuable source of motivated potential pen pals for elementary school students, university faculty members could also extend the scope of the pen pal correspondence research by focusing on the pre-service teachers. The pen pal project offers opportunities for pre-service teachers to interact with elementary students on an on-going basis. This communication could form the basis for teacher preparation projects analyzing student work samples or diagnosing student understanding as the basis for planning remedial intervention. Another interesting topic would be an examination of the pen pal writing by the pre-service teachers. Their work could be analyzed from a variety of perspectives including their understanding of mathematical content, ability to pose developmentally appropriate problems to their pen pals, ability to tailor mathematical content to students' lives to make it more concrete and meaningful, or even writing proficiency.

During the pilot work for this project the potential for this examination of the writing of the pre-service teachers became clear. Reading their letters, it quickly became apparent that pre-service teachers varied greatly in their questioning abilities. Some pre-service teachers persisted in posing yes-no questions or overly simple story problems. Some partners deluged their fourth graders with too many questions to answer during the allotted response time while others presented too few problems. As the fourth grade students complained to me about having too many, too few, too easy, or too hard questions, I encouraged them to be direct with their pen pals. "Write back and tell them that they asked you too many questions," I would advise.

Using each exchange of letters as a discrete data source about how pre-service teachers interacted with real students, university faculty members might have the resources and opportunity to recognize and address weaknesses among the pre-service students in a class, planning instruction and activities to help the pre-service teachers develop or improve these skills or understanding of specific mathematical concepts. Discussions with the class of pre-service teachers could be used to encourage them to pose open-ended questions and work to develop a dialogue through their letters. Both the elementary school teacher and university faculty member could address these topics with pre-service teachers. This pen pal project seems to be rich with opportunities for university faculty to extend the research focus to the pre-service teachers.

Related, Supporting Research

While not directly related to the application of writing strategies in mathematics instruction, better understanding of how people learn and better methods for assessing what students think or understand about a specific topic could greatly improve mathematics instruction.

This research has convinced me that understanding learning is a contentious, largely theoretical effort. I found that theories of learning are often contradictory and frustratingly removed from real life applications. I initially wondered why I had not asked myself about this topic before. How could I be a teacher and not know how people learned? I had and still have a philosophy about learning which guides me as a teacher, but it is not based upon established, widely-accepted research. Research into this topic should continue on many fronts as people seek to understand how the brain works and people learn from the chemical, neurological, psychological, cognitive science, sociolinguistics, and educational perspectives. This multi-faceted approach is necessary to help us better understand the complexity of human thought and learning.

As research emerges, it should be available in mainstream publications to ensure that as many parents and teachers have access to this information as possible. Parents and teachers have on-going opportunities to apply this knowledge as they work with children. Application of these findings could lead to greatly improved teaching and learning and may have direct implications for the use of writing in mathematics instruction.

Conclusion

I have learned a great deal about my students and their understanding of mathematical concepts through my use of writing with mathematics. Brief, regular written assessments in class have helped me identify students who need focused assistance on a topic. Using writing in this fashion has enabled me to better meet the needs of all of my students by allowing me to direct remedial assistance as needed throughout the year. My analysis of the pen pal letter exchange brought class-wide trends to my attention. These trends have influenced the curricular sequence I use and encouraged me to direct my thoughts and energies toward my students who are less confident and competent at applying mathematical concepts to situations. I hope in the future to create classroom support for these students which will help them develop the confidence and competence they need to become mathematically powerful individuals.

I plan to continue my focus on expanding the mathematics curriculum, too. I plan to persist in my efforts to include the conceptual (practical) thread of understanding mathematical concepts and contexts and the empowering (emancipatory) thread of the social applications of mathematics. Mathematics, after all, is more than arithmetic.

EPILOGUE

I began this project thinking a mathematical pen pal writing experience would be fun for my students while sneaking in both more mathematics and writing practice. I end the project believing I have been issued a challenge. Considering the work of my least mathematically successful students convinced me that I need to do more to help these students connect mathematics to the real world. I want to help these students develop skill and confidence in applying mathematical concepts to situations and recognizing the mathematics within the situations they encounter. Working with these students throughout the year, I found that most of them realized they had some difficulties with mathematics but they kept trying. I want to help these students catch up before they give up. I want them to become mathematically powerful people.

Completing this project has also renewed my enthusiasm for children's picture books with mathematical themes. These books help place mathematical concepts in everyday situations an provide visual support through their illustrations. I have read many of these books to my class, following up with class discussions and activities. My students have created their own picture books with strong mathematical content, too. Their creativity and sense of humor regularly impresses me. These projects are often quite time-consuming, however, and students complete only a few problems when they are creating a mathematical picture book. I would like to develop a way to regularly include problem-writing as part of my mathematics curriculum so that my student have many opportunities to write and sketch illustrations for problems. I am now inspired to pursue independent writing in this niche myself, too. It is time to start writing and field-testing my creations with my students!

Also, I would like to share my experience with mathematical pen pals with a broad audience of teachers. I have spoken about the project with colleagues and shared many ideas for classroom applications. I believe that writing articles for teaching magazines would allow me to share this experience and its implications with many interested teachers around the country. I think that more teachers will be open to incorporating writing into the mathematics curriculum as they learn methods for doing this and begin to perceive the benefits for their students.

Completing this research has been a roller coaster experience. The thrill of finding an idea which has been explored very little in academic circles was followed by the frustration of a seemingly endless task of reading, conducting research, reading, writing, reading, and rewriting. The excitement of conducting research in my own classroom with students with whom I felt a strong personal bond was paired with the overwhelming isolation of reading research for months. Setbacks along the way undercut my previously unshaken belief that I had the perseverance and dedication to meet any challenge I accepted. I finish with a mixture of both pride and relief.

APPENDIX ONE

SUGGESTED PEN PAL LETTER FORMAT

Sample Math Pen Pal Letter

Date

Dear Math Pen Pal,

Write a brief personal introduction (or respond to such questions from letter you just received). This can be an opportunity to share your math background with your math pen pal.

Present answers to problems posed by last letter you received (does not apply to first letter you write).

Evaluate or analyze problems (could be ones you solved or the ones you presented), complimenting specific questions or aspects of questions and making suggestions about how they could be improved. Clarify any misunderstandings. (This also does not apply to the first letter you write.)

Offer new math problem challenges to your math pen pal. Give specific directions about the kind of response you would like. (I will emphasize that they should write not only the specific answers, but also an explanation of how they arrived at them.)

Sincerely,

Math Pen Pal

PLEASE bring your first letter to class next week to be collected. You will know the name of your math pen pal when s/he writes the first reply.

APPENDIX TWO

HUMAN SUBJECTS REVIEW BOARD APPROVAL

SJSU SAN JOSE STATE UNIVERSITY

A campus of The California State University

Office of the Academic Vice President • Associate Academic Vice President • Graduate Studies and Research
One Washington Square • San Jose, California 95192-0025 • 408/924-2480

TO: Jill Troy
1641 Belleville Way, Apt. N
Sunnyvale, CA 94087

FROM: Serena W. Stanford *Serena W. Stanford*
AAVP, Graduate Studies & Research

DATE: October 31, 1996

The Human Subjects-Institutional Review Board has approved your request to use human subjects in the study entitled:

"Using Children's Writing as a Window into their Mathematical Thinking"

This approval is contingent upon the subjects participating in your research project being appropriately protected from risk. This includes the protection of the anonymity of the subjects' identity when they participate in your research project, and with regard to any and all data that may be collected from the subjects. The Board's approval includes continued monitoring of your research by the Board to assure that the subjects are being adequately and properly protected from such risks. If at any time a subject becomes injured or complains of injury, you must notify Serena Stanford, Ph.D., immediately. Injury includes but is not limited to bodily harm, psychological trauma and release of potentially damaging personal information.

Please also be advised that all subjects need to be fully informed and aware that their participation in your research project is voluntary, and that he or she may withdraw from the project at any time. Further, a subject's participation, refusal to participate, or withdrawal will not affect any services the subject is receiving or will receive at the institution in which the research is being conducted.

If you have any questions, please contact me at (408) 924-2480.

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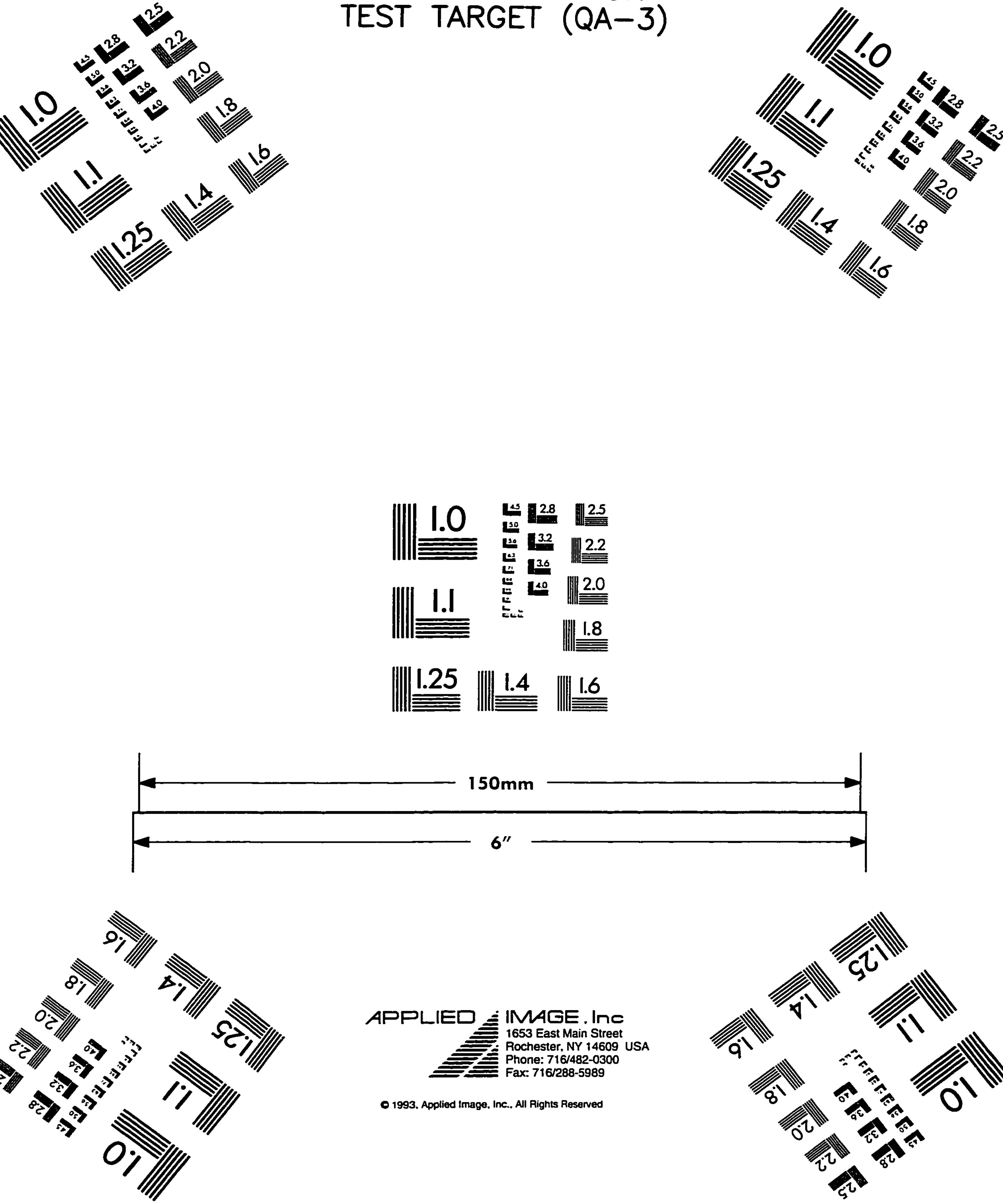
IMAGE EVALUATION TEST TARGET (QA-3)



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